ANIMATION MATHS

Ivo De Pauw and Bieke Masselis

Animation maths

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Contents

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Chapter 1 · Arithmetic refresher

As this chapter offers all the necessary mathematical skills for the full mastery of all further topics explained in this book, we strongly recommend it. To serve its purpose, the successive paragraphs below refresh some required aspects of mathematical language as used on the applied level.

1.1 Algebra

Real numbers

We typeset the set of:

- natural numbers (unsigned integers) as $\mathbb{N}$ including zero,
- integer numbers as $\mathbb{Z}$ including zero,
- rational numbers as $\mathbb{Q}$ including zero,
- real numbers (floats) as $\mathbb{R}$ including zero.

All the above make a chain of subsets: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

To avoid possible confusion, we outline a brief glossary of mathematical terms. We recall that using the correct mathematical terms reflects correct mathematical thinking. Putting down ideas in the correct words is of major importance for profound insight.

Sets

- We recall writing all **subsets** in between braces, e.g. the **empty set** appears as $\{\}$.
- We define a **singleton** as any subset containing only one element, e.g. $\{5\} \subset \mathbb{N}$, as a subset of natural numbers.
- We define a **pair** as any subset containing just two elements, e.g. $\{115, -4\} \subset \mathbb{Z}$, as a subset of integers. In programming the boolean values *true* and *false* make up a pair $\{true, false\}$ called the boolean set which we typeset as $\mathbb{B}$.
- We define $\mathbb{Z}^- = \{\ldots, -3, -2, -1\}$ whenever we need negative integers only. We express symbolically that -1234 is an **element** of $\mathbb{Z}^-$ by typesetting $-1234 \in \mathbb{Z}^-$.
- We typeset the **set minus** operator to delete elements from a set by using a backslash, e.g. $\mathbb{N} \setminus \{0\}$ reading all natural numbers except zero, $\mathbb{Q} \setminus \mathbb{Z}$ meaning all pure rational numbers after all integer values left out and $\mathbb{R} \setminus \{0, 1\}$ expressing all real numbers apart from zero and one.

Calculation basics

operation	expression	*a*	*b*	*c*
to add	$a+b=c$	term	term	sum
to subtract	$a-b=c$	term	term	difference
to multiply	$a\cdot b=c$	factor	factor	product
to divide	$\frac{a}{b}=c, b\neq 0$	numerator	divisor or denominator	quotient or fraction
to exponentiate	$a^b=c$	base	exponent	power
to take the root	$\sqrt[b]{a}=c$	radicand	index	radical
return factorial	$n!=c$			n factorial

We define the **factorial** of a natural argument as the returned product of this argument multiplied with all natural numbers from this number n down to 1. Put in symbols:

$$n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1 \quad \text{restricted to } n\in\mathbb{N}$$

Furthermore we define $1! = 1$ and as well $0! = 1$.

Examples:

$$2! = 2\cdot 1 = 2, \quad 3! = 3\cdot 2\cdot 1 = 6, \quad 4! = 4\cdot 3\cdot 2\cdot 1 = 24.$$

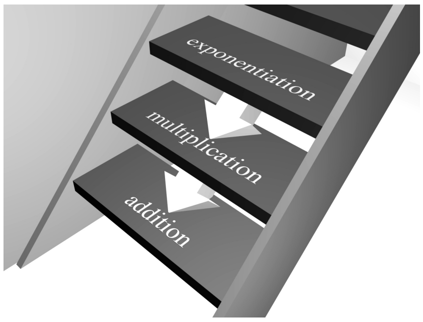
We write the **opposite** of a real number r as $-r$, defined by the sum $r+(-r)=0$. We typeset the **reciprocal** of a nonzero real number r as $\frac{1}{r}$ or r^{-1}, defined by the product $r\cdot r^{-1}=1$.

We define **subtraction** as equivalent to adding the opposite: $a-b=a+(-b)$. We define **division** as equivalent to multiplying with the reciprocal: $a:b=a\cdot b^{-1}$.

When we mix operations we need to apply priority rules for them. There is a fixed priority list ‘PEMDAS’ in performing mixed operations in $\mathbb{R}$ that can easily be memorised by ‘Please Excuse My Dear Aunt Sally’.

- First process all that is delimited in between Parentheses,
- then Exponentiate,
- then Multiply and Divide from left to right,
- finally Add and Subtract from left to right.

Now we discuss the **distributive law** ruling within $\mathbb{R}$, which we define as threading a 'superior' operation over an 'inferior' operation. In conclusion, distributing requires two *different* operations.



Hence we distribute *exponentiating* over *multiplication* as in $(a \cdot b)^3 = a^3 \cdot b^3$. Likewise rules *multiplying* over *addition* as in $3 \cdot (a+b) = 3 \cdot a + 3 \cdot b$.

However we should never stumble on this 'Staircase of Distributivity' by going too fast:

$$(a+b)^3 \neq a^3 + b^3,$$

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b},$$

$$\sqrt{x^2+y^2} \neq x + y.$$

Fractions

A **fraction** is what we call any rational number written as $\frac{t}{n}$ given $t, n \in \mathbb{Z}$ and $n \neq 0$, wherein t is called the **numerator** and n the **denominator**. We define the reciprocal of a nonzero fraction $\frac{t}{n}$ as $\frac{1}{\frac{t}{n}} = \frac{n}{t}$ or as the power $\left(\frac{t}{n}\right)^{-1}$. We define the opposite fraction as $-\frac{t}{n} = \frac{-t}{n} = \frac{t}{-n}$. We summarise fractional arithmetic:

sum	$\frac{t}{n} + \frac{a}{b} = \frac{t \cdot b + n \cdot a}{n \cdot b}$
difference	$\frac{t}{n} - \frac{a}{b} = \frac{t \cdot b - n \cdot a}{n \cdot b}$
product	$\frac{t}{n} \cdot \frac{a}{b} = \frac{t \cdot a}{n \cdot b}$
division	$\frac{\frac{t}{n}}{\frac{a}{b}} = \frac{t}{n} \cdot \frac{b}{a}$
exponentiation	$\left(\frac{t}{n}\right)^m = \frac{t^m}{n^m}$
singular fractions	$\frac{1}{0} = \pm\infty$ **infinity** (see page **??**)
	$\frac{0}{0} = ?$ **indeterminate**

Powers

We define a **power** as any real number written as g^m, wherein g is called its **base** and m its **exponent**. The opposite of g^m is simply $-g^m$. The reciprocal of g^m is $\frac{1}{g^m} = g^{-m}$, given $g \neq 0$.

According to the exponent type we distinguish between:

$$
\begin{aligned}
&g^3 = g \cdot g \cdot g && 3 \in \mathbb{N}, \\
&g^{-3} = \tfrac{1}{g^3} = \tfrac{1}{g \cdot g \cdot g} && -3 \in \mathbb{Z}, \\
&g^{\frac{1}{3}} = \sqrt[3]{g} = w \Leftrightarrow w^3 = g && \tfrac{1}{3} \in \mathbb{Q}, \\
&g^0 = 1 && g \neq 0.
\end{aligned}
$$

Whilst calculating powers we may have to:

$$
\begin{aligned}
&\text{multiply} && g^3 \cdot g^2 = g^{3+2} = g^5, \\
&\text{divide} && \tfrac{g^3}{g^2} = g^3 \cdot g^{-2} = g^{3-2} = g^1, \\
&\text{exponentiate} && \left(g^3\right)^2 = g^{3 \cdot 2} = g^6 \ \text{ them.}
\end{aligned}
$$

We insist on avoiding typesetting radicals like $\sqrt[7]{g^3}$ and strongly recommend their contemporary notation using radicand g and exponent $\frac{3}{7}$, consequently exponentiating g to $g^{\frac{3}{7}}$. We recall the fact that all square roots are non-negative numbers, $\sqrt{a} = a^{\frac{1}{2}} \in \mathbb{R}^+$ for $a \in \mathbb{R}^+$.

As well as knowing the above exponent types, understanding the above rules to calculate them is necessary for using powers successfully. We advise memorising the integer squares running from $1^2 = 1$, $2^2 = 4$, ..., up to $15^2 = 225$, $16^2 = 256$ and the integer cubes running from $1^3 = 1$, $2^3 = 8$, ..., up to $7^3 = 343$, $8^3 = 512$ in order to easily recognise them.

Recall that the only way out of any power is exponentiating with its reciprocal exponent. For this purpose we need to exponentiate both left hand side and right hand side of any given relation (see also paragraph 1.2).

Example: Find x when $\sqrt[7]{x^3} = 5$ by exponentiating this power.

$$x^{\frac{3}{7}} = 5 \Longleftrightarrow \left(x^{\frac{3}{7}}\right)^{\frac{7}{3}} = (5)^{\frac{7}{3}} \Longleftrightarrow x \approx 42.7494.$$

We emphasise the above strategy as the only successful one to free base x from its exponent, yielding its correct expression numerically approximated if we wish to.

Example: Find x when $x^2 = 5$ by exponentiating this power.

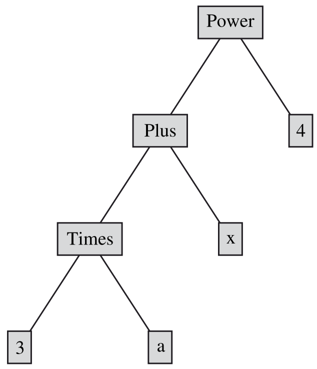
$$x^2 = 5 \Longleftrightarrow \left(x^2\right)^{\frac{1}{2}} = (5)^{\frac{1}{2}} \text{ or } -(5)^{\frac{1}{2}} \Longleftrightarrow x \approx 2.23607 \text{ or} -2.23607.$$

We recall the above double solution whenever we free base x from an *even* exponent, yielding their correct expression as accurately as we wish to.

Mathematical expressions

Composed mathematical expressions can often seem intimidating or cause confusion. To gain transparency in them, we firstly recall indexed variables which we define as subscripted to count them: $x_1, x_2, x_3, x_4, \ldots, x_{99999}, x_{100000}, \ldots$, and $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots$. It is common practice in industrial research to use thousands of variables, so just picking unindexed characters would be insufficient. Taking our own alphabet as an example, it would only provide us with 26 characters.

We define finite expressions as composed of (mathematical) operations on objects (numbers, variables or structures). We can for instance analyse the expression $(3a+x)^4$ by drawing its **tree form**. This example reveals a Power having exponent 4 and a subexpression in its base. The base itself yields a sum of the variable x Plus another subexpression. This final subexpression shows the product 3 Times a.



Let us also evaluate this expression $(3a+x)^4$. Say $a = 1$, then we see our expression partly collapse to $(3+x)^4$. If, on top of this, we assign $x = 2$, our expression then finally turns to the numerical value $(3+2)^4 = 5^4 = 625$.

When we expand this power to its **pure sum expression** $81a^4 + 108a^3x + 54a^2x^2 + 12ax^3 + x^4$, we did nothing but *reshape* its **pure product expression** $(3a+x)^4$.

We warn that trying to solve this expression – which is not a relation – is completely in vain. Recall that inequalities, equations and systems of equations or inequalities are the only objects in the universe we can (try to) solve mathematically.

Relational operators

We also refresh the use of correct terms for inequalities and equations.

We define an **inequality** as any *variable* expression comparing a left hand side to a right hand side by applying the 'is-(strictly)-less-than' or by applying the 'is-(strictly)-greater-than' operator. For example, we can read $(3a+x)^4 \leqslant (b+4)(x+3)$ containing variables *a, x, b*. Consequently we may solve such inequality for any of the unknown quantities a, x or b.

We define an **equation** as any *variable* expression comparing a left hand side to a right hand side by applying the 'is-equal-to' operator. For example $(3a+x)^4 = (b+4)(x+3)$ is an equation containing variables *a, x, b*. Consequently we also may solve equations for any of the unknown quantities a, x or b.

We define an **equality** as a constant relational expression being *true*, e.g. $7 = 7$. We define a **contradiction** as a constant relational expression being *false*, e.g. $-10 > 5$.

Real polynomials

We elaborate upon the mathematical environment of polynomials over the real numbers in their variable or indeterminate x, a set we denote with $\mathbb{R}_{[x]}$.

- Monomials

  We define a **monomial** in x as any product ax^n, given $a \in \mathbb{R}$ and $n \in \mathbb{N}$. We can extend this concept to several indeterminates $x, y, z, \ldots$ like the monomials $3(xy)^6$ and $3(x^2y^3z^6)$ are.

  We define the **degree** of a monomial ax^n as its natural exponent $n \in \mathbb{N}$ to the **indeterminate part** x. We say constant numbers are monomials of degree 0 and linear terms are monomials of degree 1. We say squares have degree 2 and cubes have degree 3, followed by monomials of higher degree.

  For instance, the real monomial $-\sqrt{12}x^6$ is of degree 6. Extending this concept, the monomial $3(xy)^6$ is of degree 6 in xy and the monomial $3(x^2y^3z^6)^9$ is of degree 9 in $x^2y^3z^6$.

  We define **monomials of the same kind** as those having an identical indeterminate part. For instance, both $\frac{5}{7}x^6$ and $-\sqrt{12}x^6$ are of the same kind. Extending the concept, likewise $\frac{5}{7}x^3y^5z^2$ and $-\sqrt{12}x^3y^5z^2$ are of the same kind.

  All basic operations on monomials emerge simply from applying the calculation rules of fractions and powers.

- Polynomials

  We define a **polynomial** $V(x)$ as any sum of monomials. We define the **degree** of $V(x)$ as the maximal exponent $m \in \mathbb{N}$ to the indeterminate variable x. For instance, the real polynomial

  $$V(x) = 17x^2 + \frac{1}{4}x^3 + 6x - 7x^2 - \sqrt{12}x^6 - 13x - 1,$$

  is of degree 6.

  Whenever monomials of the same kind appear in it, we can simplify the polynomial. For instance, our polynomial simplifies to $V(x) = 10x^2 + \frac{1}{4}x^3 - 7x - \sqrt{12}x^6 - 1$.

  Moreover, we can sort any given polynomial either in an ascending or descending order according to its powers in x. Sorting our polynomial $V(x)$ in an ascending order yields $V(x) = -1 - 7x + 10x^2 + \frac{1}{4}x^3 - \sqrt{12}x^6$. Sorting $V(x)$ in a descending order yields $V(x) = -\sqrt{12}x^6 + \frac{1}{4}x^3 + 10x^2 - 7x - 1$.

Eventually we are able to evaluate any polynomial, getting a numerical value from it. For instance evaluating $V(x)$ in $x=-1$, yields $V(-1) = -\sqrt{12}(-1)^6 + \frac{1}{4}(-1)^3 + 10(-1)^2 - 7(-1) - 1 = -\sqrt{12} - \frac{1}{4} + 16 = \frac{63}{4} - 2\sqrt{3} \in \mathbb{R}$.

▷ Basic operations

Adding two monomials of the same kind: we add their coefficients and keep their indeterminate part

$$5a^2 - 3a^2 = (5-3)a^2 = 2a^2.$$

Multiplying two monomials of any kind: we multiply both their coefficients and their indeterminate parts

$$-5ab \cdot \frac{7}{4}a^2b^3 = -5 \cdot \frac{7}{4} \cdot a^{1+2}b^{1+3} = \frac{-35}{4}a^3b^4.$$

Dividing two monomials: we divide both their coefficients and their indeterminate parts

$$\frac{-8a^6b^4}{-4a^4} = \frac{-8}{-4}\, a^{6-4}b^{4-0} = 2a^2b^4.$$

Exponentiating a monomial: we exponentiate each and every factor in the monomial

$$\left(-2a^2b^4\right)^3 = (-2)^3(a^2)^3(b^4)^3 = -8a^6b^{12}.$$

Adding or subtracting polynomials: we add or subtract all monomials of the same kind

$$(x^2 - 4x + 8) - (2x^2 - 3x - 1) = x^2 - 4x + 8 - 2x^2 + 3x + 1 = -x^2 - x + 9.$$

Multiplying two polynomials: we multiply each monomial of the first polynomial with each monomial of the second polynomial and simplify all those products to the resulting product polynomial

$$\begin{aligned}(2x^2 + 3y) \cdot (4x^2 - y) &= 2x^2(4x^2 - y) + 3y(4x^2 - y) \\ &= 2x^2 \cdot 4x^2 + 2x^2 \cdot (-y) + 3y \cdot 4x^2 \\ &\quad\quad + 3y \cdot (-y) \\ &= 8x^4 - \boxed{2x^2y} + \boxed{12x^2y} - 3y^2 \\ &= 8x^4 + 10x^2y - 3y^2.\end{aligned}$$

1.2 Equations in one variable

In anticipation of this section, we will refresh the required vocabulary. A **solution** is any value assigned to the variable that turns the given equation into an *equality* (being *true*). The **scope** of an equation is any number set in which the equation resides, realising it will most likely be $\mathbb{R}$. We define the **solution set** as the set containing all legal solutions to an equation. This solution set is always a subset of the scope of the equation.

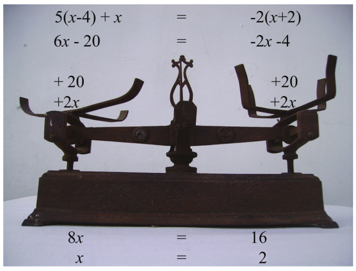
Linear equations

A **linear equation** is an algebraic equation of degree one, referring to the maximum natural exponent of the unknown quantity. By simplifying we can always standardise any linear equation to

$$ax + b = 0, \tag{1.1}$$

given $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$. We cite $3x+7=22, 5x-9d=c$ and $5(x-4)+x=-2(x+2)$ as examples of linear equations, and $3x^2+7=22$ and $5ab-9b=c$ as counter examples. The adjective 'linear' originates from the Latin word 'linea' meaning (straight) line as referring to the graph of a linear function (see chapter **??**).

We solve a linear equation for its unknown part by rewriting the entire equation until its shape exposes the solution explicitly.



We recall easily the required rules for rewriting a linear equation by the metaphor denoting a linear equation as a 'pair of scales'. This way we should never forget to keep the equation's balance: whatever operation we apply, it has to act on both sides of the equals-sign. If we add to (or subtract from) the left hand 'scale' than we are obliged to add the same term to (or subtract it from) the right hand 'scale'. If we multiply (or divide) the left hand side, than we are likewise obliged to multiply (or divide) the right hand side with the same factor. If not, our equation would lose its balance just like a pair of scales would. We realise that our metaphor covers all usual 'rules' to handle linear equations.

The reason we perform certain rewrite steps depends on which variable we are aiming for. This is called *strategy*. Solving the equation for a different variable implies a different sequence of rewrite steps.

Example: We solve the equation $5(x-4)+x=-2(x+2)$ for x. Firstly, we apply the distributive law: $5x-20+x=-2x-4$. Secondly, we put all terms dependent of x to the left hand side and the constant numbers to the right hand side $5x+x+2x=-4+20$. Thirdly, we simplify both sides $8x=16$. Finally, we find $x=2$ leading to the solution singleton $\{2\}$.

Quadratic equations

Handling quadratic expressions and solving quadratic equations are useful basics in order to study topics in multimedia, digital art and technology.

▷ Expanding products

We refresh **expanding** a product as (repeatedly) applying the distributive law until the initial expression ends up as a pure *sum* of terms. Note that our given polynomial $V(x)$ itself does not change: we just shift its appearance to a pure sum. We illustrate this concept through $V(x)=(2x-3)(4-x)$.

$$\begin{aligned}(2x-3)(4-x) &= (2x-3)\cdot 4+(2x-3)\cdot(-x)\\ &= (8x-12)+(-2x^2+3x)\\ &= -2x^2+11x?12.\end{aligned}$$

Other examples are

$$5a(2a^2-3b)=5a\cdot 2a^2-5a\cdot 3b=10a^3-15ab$$

and

$$\begin{aligned}4\left(x-\frac{1}{2}\right)\left(x+\frac{13}{2}\right) &= (4x-2)\left(x+\frac{13}{2}\right)\\ &= (4x-2)\cdot x+(4x-2)\cdot\frac{13}{2}\\ &= 4x^2-2x+26x-13=4x^2+24x-13.\end{aligned}$$

▷ Factoring polynomials

We define **factoring** a polynomial as decomposing it into a pure *product* of (as many as possible) factors. Note that our given polynomial $V(x)$ itself does not change: we just shift its appearance to a pure product. Our **trinomial** $V(x)=-2x^2+11x-12$ just shifts its appearance to the pure product $V(x)=(2x-3)(4-x)$ when factored. It merely shows that the product $(2x-3)(4-x)$ is a factorisation of the trinomial $-2x^2+11x-12$.

Imagine we had to factor the trinomial $-2x^2+11x-12$ without any hint. This way we realise that factoring generally is a hard job to do. Especially because we do not have any clue about which factors build up the pure product for a polynomial. Many questions arise: *how many* factors to expect, *where* to start from, *what* is the opening step towards factorisation?

We observe the need for at least a minimum asset of factoring methods. As an extra motivation, we emphasise the importance of factoring as it reveals all essential building blocks of any polynomial. Knowing the roots of a polynomial gives us a deeper insight. We therefore introduce some factoring basics in the next paragraphs.

Common Factor

We show how to separate common factors if they appear.

For instance $6+12x = 6\cdot 1+6\cdot(2x) = 6\cdot(1+2x)$ results in a pure product of a number and a linear factor by separating the common factor 6. Another polynomial like $5x+x^2 = 5x+xx = (5+x)\cdot x$ separates into two linear factors by the use of the common factor x. An example expression like $39x+3xy = 3\cdot 13x+3xy = 3\cdot x\cdot(13+y)$ yields a pure product of a number factor, a linear factor in x and a linear factor in y by separating the common factors 3 and x. Occasionally we may have to factor by grouping. For instance

$$\begin{aligned} 1+x+x^2+x^3 &= (1+x)+(x^2+x^3) = (1+x)+(x^2\cdot 1+x^2\cdot x) \\ &= (1+x)+x^2(1+x) = 1\cdot(1+x)+x^2(1+x) \\ &= (1+x^2)\cdot(1+x) \end{aligned}$$

results stepwise into a pure product of a quadratic and a linear factor in x.

Perfect powers

Expanding the natural powers of the **binomial** $A+B$ reveals their corresponding pure sum shapes.

$$\begin{aligned} (A+B)^2 &= (A+B)\ (A+B) = A^2+2AB+B^2 \\ (A+B)^3 &= (A+B)^2(A+B) = A^3+3A^2B+3AB^2+B^3 \\ (A+B)^4 &= (A+B)^3(A+B) = A^4+4A^3B+6A^2B^2+4AB^3+B^4 \end{aligned}$$

We define a **perfect power** as any natural exponentiation of a binomial. The important power is $(A+B)^2$ which we define as the **perfect square** of its binomial $A+B$.

Those perfect powers of $A+B$, when ordered to ascending natural exponents, display **Pascal's Triangle** for all $n \in \mathbb{N}$.

$$\begin{gathered} 1 \\ 1A+1B \\ 1A^2+2AB+1B^2 \\ 1A^3+3A^2B+3AB^2+1B^3 \\ 1A^4+4A^3B+6A^2B^2+4AB^3+1B^4 \\ 1A^5+5A^4B+10A^3B^2+10A^2B^3+5AB^4+1B^5 \\ \vdots \end{gathered}$$

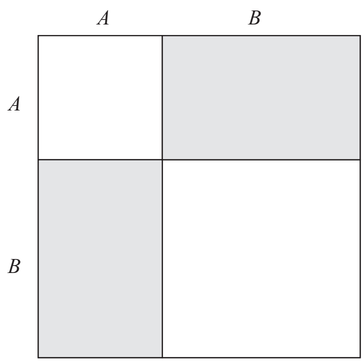
Notice how a **coefficient** is produced as a sum of its upper two, leading to a symmetric triangle of numbers with the constant '1' on both edges. This 'triangle' is named after its explorer **Blaise Pascal** (1623 –1662).

Despite the diminishing need for perfect power formulas in this century of ruling computing power, we do advise you to know at least the perfect square by heart.

To put the perfect square into words: *'The square of a binomial equals the sum of both squares plus two times the product'*.

$$(A+B)^2 = A^2 + 2AB + B^2. \tag{1.2}$$

We provide a visual aid to help you memorise it. The area of the total square equals $(A+B)\cdot(A+B)$. Alternatively we puzzle this area piece by piece, via adding both white square areas A^2 and B^2 plus the *two* grey rectangular areas AB, jointly equalling the perfect square as the trinomial A^2+B^2+2AB.



Consequently we now can explore a new factoring method. For instance, we intend to factor the trinomial $1-2x+x^2$, whilst we have no guarantee for its pure product shape to even exist.

Strategically we perform two subsequent checks.

1) Verify whether both squares carry the same sign.
2) Then find $2AB$ corresponding correctly to the given A and B.

Only when both checks hold, are we able to shift the given trinomial to its perfect product $(A+B)^2$. We give an example of this strategy to the trionomial $1-2x+x^2$.

We rewrite the trinomial to $+(-1)^2 - 2x + (x)^2$ by assigning $A = -1$ and $B = x$. By substituting A and B into $2AB$, we find $+(-1)^2 + 2(-1)(x) + (x)^2$ equalling our trinomial. Therefore we confirm $A = -1$ and $B = x$, which allows the shift $1 - 2x + x^2 = (-1 + x)^2$. We realise that alternatively $(+1 - x)^2$ is a correct factorisation as well.

Perfect quotient

$$(A+B)(A-B) = A^2 - B^2 \tag{1.3}$$

- Quadratic formula for quadratic equations

A **quadratic equation** is an algebraic equation of degree two in the unknown quantity x that can be reduced to the default shape

$$ax^2 + bx + c = 0 \tag{1.4}$$

given $a \in \mathbb{R} \setminus \{0\}$ and $b, c \in \mathbb{R}$.

To solve this equation for x we firstly divide both sides of it by a . Dividing by a is valid since $a \neq 0$. In case of $a = 0$ we would no longer have a quadratic but a linear equation.

$$ax^2 + bx + c = 0 \Longleftrightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Secondly, we aim for a perfect square by adding and subtracting the special term $\left(\frac{b}{2a}\right)^2$ which is again valid since this is equivalent to adding 0. This way we have created a perfect square $(A+B)^2$, assigning $A = x$ and $B = \frac{b}{2a}$.

$$\begin{aligned}
& x^2 + \mathbf{2}\left(\frac{b}{\mathbf{2}a}\right)x + \left(\frac{\mathbf{b}}{\mathbf{2a}}\right)^{\mathbf{2}} - \left(\frac{\mathbf{b}}{\mathbf{2a}}\right)^{\mathbf{2}} + \frac{c}{a} = 0 \\
\Longleftrightarrow & \left(x^2 + \mathbf{2}\left(\frac{b}{\mathbf{2}a}\right)x + \left(\frac{\mathbf{b}}{\mathbf{2a}}\right)^{\mathbf{2}}\right) - \left(\frac{\mathbf{b}}{\mathbf{2a}}\right)^{\mathbf{2}} + \frac{c}{a} = 0 \\
\Longleftrightarrow & \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\
\Longleftrightarrow & \left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}
\end{aligned}$$

The left hand side of this equation is now a square. Before proceeding we make sure that all denominators of the right hand side are equal.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c \cdot 4a}{a \cdot 4a} \Longleftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Arithmetically this holds: $L^2 = R \Leftrightarrow L = \sqrt{R}$ or $L = -\sqrt{R}$. Hence we reach two similar solutions to our equation:

$$\begin{array}{ccc} x + \dfrac{b}{2a} = \sqrt{\dfrac{b^2-4ac}{4a^2}} & \text{or} & x + \dfrac{b}{2a} = -\sqrt{\dfrac{b^2-4ac}{4a^2}} \\ & \Downarrow & \\ x + \dfrac{b}{2a} = \dfrac{\sqrt{b^2-4ac}}{2a} & \text{or} & x + \dfrac{b}{2a} = -\dfrac{\sqrt{b^2-4ac}}{2a} \\ & \Downarrow & \\ x = -\dfrac{b}{2a} + \dfrac{\sqrt{b^2-4ac}}{2a} & \text{or} & x = -\dfrac{b}{2a} - \dfrac{\sqrt{b^2-4ac}}{2a} \end{array}$$

The **discriminant** of a quadratic equation $ax^2 + bx + c = 0$, given $a \neq 0$, is the real number to be calculated as

$$D = b^2 - 4ac. \tag{1.5}$$

Furthermore, we solve $ax^2 + bx + c = 0$ for x like this:

if $D < 0$ then there are no solutions in $\mathbb{R}$,

if $D = 0$ we find one real root $x_1 = \frac{-b}{2a}$,

if $D > 0$ we have two similar **roots**

$$x_1 = \frac{-b+\sqrt{D}}{2a} \text{ and } x_2 = \frac{-b-\sqrt{D}}{2a}. \tag{1.6}$$

As a spin-off these roots enable factoring the default left hand side as

$$ax^2 + bx + c = a(x - x_1)(x - x_2) \quad \text{when } D > 0 \tag{1.7}$$

and as

$$ax^2 + bx + c = a(x - x_1)^2 \quad \text{when } D = 0.$$

Examples: Solving the quadratic equation $-2x^2 + 11x - 12 = 0$ for x, we firstly calculate its discriminant $D = 11^2 - 4 \cdot (-2) \cdot (-12) = 25$ to subsequently determine its roots as $x_1 = \frac{-11+\sqrt{25}}{2\cdot(-2)} = \frac{3}{2}$ and $x_2 = \frac{-11-\sqrt{25}}{2\cdot(-2)} = 4$. As a bonus, this allows us to factor $-2x^2 + 11x - 12$ as $(-2)\left(x - \frac{3}{2}\right)(x-4)$. The solution set is the pair $\{\frac{3}{2}, 4\} \subset \mathbb{R}$.

We solve $25x^2 - 60x + 36 = 0$ for x as a next example. In this case the discriminant equals zero, yielding a unique root of multiplicity 2 to be found as $x = \frac{-(-60)\pm\sqrt{0}}{2\cdot 25} = \frac{6}{5}$ and thus leading to the solution singleton $\{\frac{6}{5}\} \subset \mathbb{R}$.

Finally solving also $25x^2 + 49x + 36 = 0$ for x, we calculate its discriminant as $D = -1199$. It is not possible for us to find any real root due to the fact $\sqrt{-1199} \notin \mathbb{R}$, which in this case leads to an empty solution set $\{\} \subset \mathbb{R}$.

Equations of higher degree

Solving the polynomial quadratic equation $ax^2+bx+c=0$ for x by means of the Quadratic Formula

$$\frac{-b\pm\sqrt{D}}{2a}$$

dates back to Babylonian and Greek times. The next big leap forward for solving equations of higher degree had to wait until the 16$^{\text{th}}$ century, till the time of the Renaissance.

- ▷ Cubic equations

  Geronimo Cardano (1501–1576) published a similar Cubic Formula for solving polynomial equations of degree three. Despite Cardano publishing it, the Cubic Formula was actually discovered by another Italian. Historians claim that this formula was discovered by the mathematician **Niccolo Fontana** (1499 –1557) (nicknamed Tartaglia or '*stutterer*').

- ▷ Quartic equations

  Shortly after the former formula, **Lodovico Ferrari** (1522 – 1565) a pupil of Cardano, and also an Italian mathematician, found the Quartic formula to solve polynomial equations of degree four.

- ▷ Quintic equations

  For an apotheosis one needed to wait until the 19$^{\text{th}}$ century in France: the very young and brilliant mathematician **Evariste Galois** (1811–1832) proved the impossibility of finding a similar Quintic Formula for polynomial equations of larger than degree four. Meanwhile, as a workaround (from, among others, Isaac Newton, around 1676) we can solve any polynomial equation numerically, yielding approximations for its solutions. Apart from this modern numerical approach, special subtypes of polynomial equations of larger degree can also still be solved exactly by means of formulas of radical expressions.

1.3 Logarithms

- ▷ We define the **common or Briggsian logarithm** as an exponent to base 10,

$$\log_{10}(a) = x \Leftrightarrow 10^x = a$$

which satisfies the existence condition $a\in\mathbb{R}^+\setminus\{0\}$ with base $10\in\mathbb{R}^+\setminus\{0,1\}$.

Examples:

$$\begin{aligned}\log_{10}(100) &= 2\\ \log_{10}(1000) &= 3\\ \log_{10}(100000) &= 5\end{aligned}$$

The decimal logarithm is an idea of the Englishman **Henry Briggs** (1561–1630), contemporary of the Scotsman **John Napier** (1550 – 1617). Around 1615, both mathematicians agreed that the logarithm base number 10 would offer the better future perspective. Among others, the famous scientist **Simon Stevin** (1548 – 1620) from Bruges, in Belgium, contributed a lot in establishing decimal numbers worldwide, in line with base number 10. In 1624, Briggs published the very first decimal logarithm table in his book 'Arithmetica Logaritmica'.

Back in 1618, Napier published (unknowingly) the **natural base**, which was later re-discovered as the transcendental limit value

$$2.718281828459\ldots \tag{1.8}$$

by the Swiss **Jakob Bernoulli** (1654 –1705) and – similarly to the transcendental number $\pi \approx 3.14$ for circles – by the next Swiss genius **Leonhard Euler** (1707–1783) named Euler's number, simply called by the symbol $e \approx 2.72$.

▷ We define the **natural logarithm** or **Napier's logarithm** as an exponent to base $e \approx 2.72$,

$$\log_e(a) = x \Leftrightarrow e^x = a$$

which satisfies the existence condition $a \in \mathbb{R}^+ \setminus \{0\}$ with base $e \in \mathbb{R}^+ \setminus \{0,1\}$.

Examples:

$$\begin{aligned}\log_e\left(e^1\right) &= 1\\ \log_e\left(e^2\right) &= 2\\ \log_e(e^5) &= 5\end{aligned}$$

John Napier conceived of the natural logarithm around 1594. After decades of calculation, he finally published the first natural logarithm table in his book 'Mirifici Logarithmorum Canonis Descriptio' in 1614. New mathematical ideas acquire a common status proportional to their ease of use, but Napier's first design based on $\frac{1}{e} \approx 0.368$ was seemingly less practical. In general, Napier contributed substantially to the popularity and adoption of decimal numbers and the decimal logarithm.

▷ We define the **binary logarithm** as an exponent to base 2,

$$\log_2(a) = x \Leftrightarrow 2^x = a$$

which satisfies the existence condition $a \in \mathbb{R}^+ \setminus \{0\}$ with base $2 \in \mathbb{R}^+ \setminus \{0,1\}$.

Examples:

$$\begin{aligned} \log_2(4) &= 2 \\ \log_2(8) &= 3 \\ \log_2(32) &= 5 \\ \log_2(1024) &= 10 \end{aligned}$$

This binary logarithm is especially applicable in data communication and other binary environments.

Annex A · Real numbers in computers

Real numbers are stored identically into the computer. From the irrational numbers such as π to giant integers such as 10^{10}, radicals and negative fractions, they all fit into the machine in the same way. Let us also add -1 billion to our example list.

A.1 Scientific notation

The storage of real numbers into computers is based on their **scientific notation** which separates the sign and the precision from the order of magnitude of each exact number x, arranged into the product

$$x = (-1)^s \times N_{10} \times 10^{E_{10}}.$$

The first factor $(-1)^s$ shows the **sign** of x, the second factor N_{10} is the decimal **normalised significand** lying between 1 and 10 and finally the exponent E_{10} indicates the **decimal order of magnitude** of x.

exact value x	decimally displayed	decimally scientifically displayed
$\frac{1}{10}$	0.1	$+1. \times 10^{-1}$
π	$3.141592653\ldots$	$+3.141592653\ldots \times 10^{0}$
0.00001234	0.00001234	$+1.234 \times 10^{-5}$
-1 billion	$-1000000000.$	$-1.000000000 \times 10^{9}$

This normalised scientific notation allows us to simulate the storage of our real examples into a decimal machine which allocates a standardised digit sequence for each of them.

A.2 The decimal computer

Let us straightforwardly consider a decimal computer which stores one digit denoting the sign, one digit indicating the order of magnitude and stores four **significant digits** of the original value x.

scientific notation	uniform machine precision	stored **machine number** x'
$+1. \qquad\qquad \times 10^{-1}$	$(-1)^0 \times 1. \qquad \times 10^{-1}$	$(-1)^0 \times 1.000 \times 10^{-1}$
$+3.141592653\ldots \times 10^0$	$(-1)^0 \times 3.142 \times 10^0$	$(-1)^0 \times 3.142 \times 10^0$
$+1.234 \qquad\quad \times 10^{-5}$	$(-1)^0 \times 1.234 \times 10^{-5}$	$(-1)^0 \times 1.234 \times 10^{-5}$
$-1.000000000 \quad \times 10^9$	$(-1)^1 \times 1.000 \times 10^9$	$(-1)^1 \times 1.000 \times 10^9$

Our simplified decimal computer stores exact values $x \in \mathbb{R}$ systematically in a fixed digit sequence x' containing the sign (1 digit), the exponent (1 digit) and the normalised significand (4 digits). This computer is limited to storing only four significant digits and consequently standardises its stored numbers x' with fixed **machine precision**. We call the finite subset of real numbers x' which are inevitably **rounded** to fit into the computer, **machine numbers**. The accompanying figure shows all positive machine numbers, from the smallest to the largest one in $\mathbb{R}^+$ in case of 8-bit (which means 2-decimal digit) numbers.

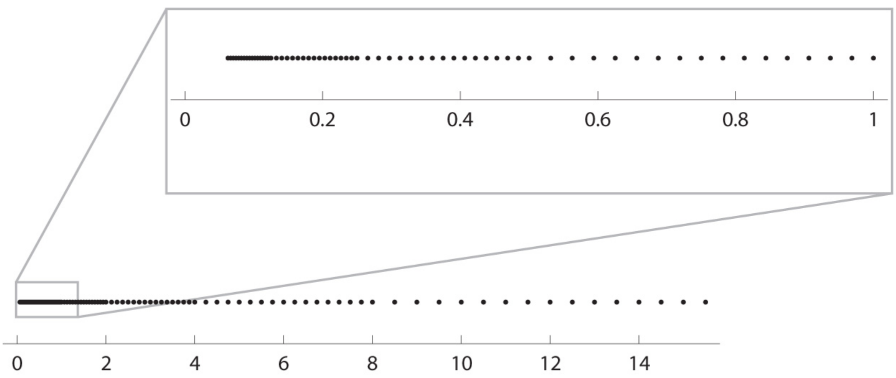


Figure A.1: The subset of (fictitious) 8-bit machine numbers x' in $\mathbb{R}^+$

A.3 Special values

Calculations which result in numbers smaller than the smallest machine number, suffer **real underflow**. Arithmetical outputs which are larger than the largest machine number feature **real overflow**. For instance, storing the real number *zero* is an issue, since it would require the exponent $E_{10} = -\infty$. To be able to store the number zero, and similarly the *infinities* requiring exponent $E_{10} = +\infty$ and the indeterminate such as $\frac{0}{0}$ in our decimal machine, we predefine these exceptions respectively as **NULL**, **INFINITY** and **NAN** abbreviating 'Not A Number'.

Annex B · Notations and Conventions

B.1 Alphabets

Latin alphabet

meaning	symbol
constants and coefficients	$a, b, c, \ldots$
unknown quantities and variables	$x, y, z, \ldots$
points	$P, Q, R, \ldots$
lines	$r, s, t, \ldots$
planes	$v_R, v_P, \ldots$
vectors	$\vec{v}, \vec{w}, \ldots$
unit vectors	$\hat{v}, \hat{w}, \ldots$
matrices	$A, B, C, \ldots$
angles	$\hat{A}, \hat{B}, \hat{C}, \ldots$
(angles alternatively in Greek)	$\alpha, \beta, \theta, \ldots$
Bezier segment	$\vec{b}_{012\ldots n}$
B-spline	$\vec{s}_{012\ldots n}$
translations	$T_{\vec{c}}$
standard rotations	R_O
standard scalings	S_O
composite action transformations	A
pivot transformation	P_B
conventional composite transformations	TRS
(nonconventional) orbit transformation	O_B
embedding transformation	E_i
camera transformation	$F_{\vec{c}}$
view transformation	$V_{\vec{c}}$
quaternions	$q, p, \ldots$
normalized quaternions	$q_n, p_n, \ldots$
unit quaternions	u
rotation quaternions	$u, r, \ldots$

Greek alphabet

Traditionally, we use Greek characters to denote angles (especially in trigonometry). We also choose Greek characters for typesetting mathematical and physical constants.

name	Greek character	name	Greek character
alpha	α	nu	ν
beta	β	xi	ξ
gamma	γ	omicron	o
delta	δ, Δ	pi	π
epsilon	ε	rho	ρ
zeta	ζ	sigma	σ
eta	η	tau	τ
theta	θ	upsilon	υ
iota	ι	phi	ϕ, Φ
kappa	κ	chi	χ
lambda	λ	psi	ψ
mu	μ	omega	ω

B.2 Mathematical symbols

Sets

number sets including zero	symbol
natural numbers (unsigned integers)	$\mathbb{N}$
integer numbers (integers)	$\mathbb{Z}$
rational numbers or fractions	$\mathbb{Q}$
real numbers (floating points)	$\mathbb{R}$
complex numbers	$\mathbb{C}$
hypercomplex numbers or quaternions	$\mathbb{H}$

We embed these number sets as

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H}.$$

Mathematical symbols

name	symbol
empty set	$\{\}$
set minus	$\setminus$
element of	$\in$
cardinality (number of elements)	#
factorial	!
equal to	$=$
equivalent with	$\Leftrightarrow$
implies	$\Rightarrow$
distance	d
difference	Δ
degrees	$^\circ$
infinity (unbound large value)	∞
summation	Σ
dot product	$\cdot$
cross product	$\times$
transpose	^{T}
conjugate	*
imaginary unities	i, j, k
cartesian coordinates	$(\ )_{\mathrm{cc}}$
polar coordinates	$(\ )_{\mathrm{pc}}$
tiny error	ε

Mathematical keywords

name	symbol
logarithm in base b	$\log_b$
exponential in base e	exp
radian	rad
sine	sin
cosine	cos
tangent	tan
cotangent	cot
arcsine	arcsin
arccosine	arccos
arctangent	arctan
extended arctangent	atan2
determinant	det
absolute value	abs

Numbers

name	symbol, (rounded) value
pi	$\pi \approx 3.1416$
radian	$1 \text{ rad} \approx 57.30^\circ$
silver number	$\delta = 1 + \sqrt{2} \approx 2.4142$
golden number	$\Phi = \frac{1+\sqrt{5}}{2} \approx 1.6180$
paired golden number	$\Phi' = \frac{1-\sqrt{5}}{2} \approx -0.6180$
imaginary unities (quaternions)	$i^2 = j^2 = k^2 = -1$ and $ij = k$
natural base	$e \approx 2.7183$
acceleration due to gravity (average)	$g \approx 9.8067$

Annex C · The International System of Units (SI)

C.1 SI Prefixes

We may use the international default prefixes to specify decimal orders of magnitude.

name	symbol	factor
yotta	Y	10^{24}
zetta	Z	10^{21}
exa	E	10^{18}
peta	P	10^{15}
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
hecto	h	10^{2}
deca	da	10^{1}
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}
zepto	z	10^{-21}
yocto	y	10^{-24}

Examples:

$$\begin{aligned} 12\ \mathrm{k}m &= 12 \times 10^{3}\ m = 12\,000\ m \\ 34\ \mathrm{m}m &= 34 \times 10^{-3}\ m = 0.034\ m \end{aligned}$$

In our modern world, we quantify measures standardised by the **SI** (the International System of Units). Nature's base measures length l, mass m and time t are measured in metres m, kilograms kg and seconds s respectively. We may typeset units by putting *square brackets* around their corresponding measures.

C.2 SI Base measures

We mainly use just these three base measures throughout this book; for the few remaining base measures we refer you to the physics literature.

measure	symbol	SI-unit	
length	l	$[l] = m$	metre
mass	m	$[m] = kg$	kilogram
time	t	$[t] = s$	second

C.3 SI Supplementary measure

Unlike the real physics units, expressing plane angles in radian is typeset by the supplementary measure or mathematical tag 'rad'.

measure	symbol	SI-unit	
plane angle	α	$[\alpha] = \text{rad}$	radian

C.4 SI Derived measures

Derived measures are composed of the above measures. We mainly use the following derived measures throughout this book; for the remaining ones with special names we refer you to the physics literature.

measure	symbol	SI-unit	
width	b	$[b] = m$	metre
height	h	$[h] = m$	metre
radius	r	$[r] = m$	metre
diameter	d	$[d] = m$	metre
distance	d	$[d] = m$	metre
norm of a location vector	$\|\vec{s}\|$	$[\|\vec{s}\|] = m$	metre
area	$area$	$[area] = m^2$	square metre
volume	$volume$	$[volume] = m^3$	cubic metre
speed	v	$[v] = \frac{m}{s}$	metre per second
magnitude of acceleration	a	$[a] = \frac{m}{s^2}$	metre per second squared
acceleration due to gravity	g	$[g] = \frac{m}{s^2}$	metre per second squared
frequency	f	$[f] = s^{-1}$	Hertz
angular location	θ	$[\theta] = \text{rad}$	radian
angular speed	ω	$[\omega] = \frac{\text{rad}}{s}$	radians per second
angular acceleration	α	$[\alpha] = \frac{\text{rad}}{s^2}$	radians per second squared

Annex D · Real numbers in computers

Real numbers are stored identically into the computer. From the irrational numbers such as π to giant integers such as 10^{10}, radicals and negative fractions, they all fit into the machine in the same way. Let us also add -1 billion to our example list.

D.1 Scientific notation

The storage of real numbers into computers is based on their **scientific notation** which separates the sign and the precision from the order of magnitude of each exact number x, arranged into the product

$$x = (-1)^s \times N_{10} \times 10^{E_{10}}.$$

The first factor $(-1)^s$ shows the **sign** of x, the second factor N_{10} is the decimal **normalised significand** lying between 1 and 10 and finally the exponent E_{10} indicates the **decimal order of magnitude** of x.

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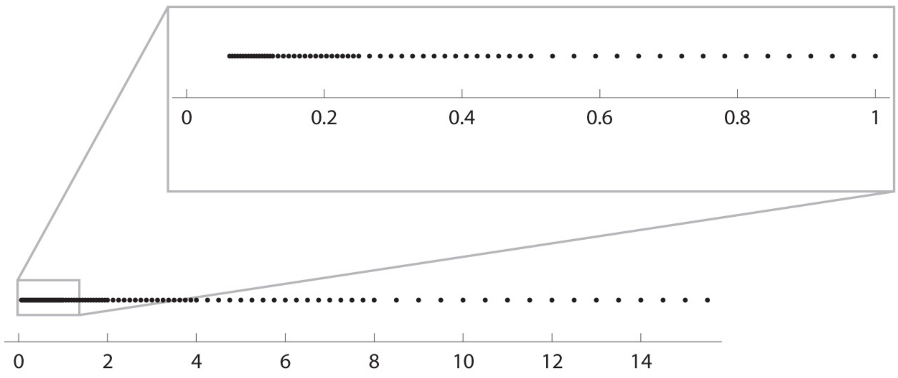


Figure D.1: The subset of (fictitious) 8-bit machine numbers x' in $\mathbb{R}^+$

D.3 Special values

Calculations which result in numbers smaller than the smallest machine number, suffer **real underflow**. Arithmetical outputs which are larger than the largest machine number feature **real overflow**. For instance, storing the real number *zero* is an issue, since it would require the exponent $E_{10} = -\infty$. To be able to store the number zero, and similarly the *infinities* requiring exponent $E_{10} = +\infty$ and the indeterminate such as $\frac{0}{0}$ in our decimal machine, we predefine these exceptions respectively as **NULL**, **INFINITY** and **NAN** abbreviating ‘Not A Number’.

Annex E · Animation Maths (2021) Answers

1. Arithmetic Refresher

Exercise 1

1) a^{10}
2) $2a^5$
3) 0
4) a^{15}
5) a^7
6) a^6
7) a^8
8) $\frac{1}{4}a^6$

Exercise 2

1) $-a-b$
2) $\frac{a+b}{-c}$ en $\frac{-a-b}{c}$
3) $\frac{4c+3d}{4d}$
4) $\frac{3c}{4d}$
5) -7
6) a^{2+n+m}
7) $-8a^6b^9$
8) c^4d^2
9) $\frac{b^2}{a^{10}}$
10) a^9

Exercise 3

1) -6
2) $\frac{26}{5}$

Exercise 4

The odd numbers are 45, 47 and 49.

Exercise 5

1) a^{-12}
2) $\frac{1}{a}$
3) a^{-20}
4) $\frac{a^6}{b^9}$
5) b^8
6) $16a^{12}b^8$
7) b^{50}
8) a^{23}
9) $-16a^6b$
10) $144a^{10}b^{20}$

Exercise 6

1) 0
2) $-16x^4y^4$
3) -1
4) $-x^{12}y^6$
5) $-x^{24}y^6$
6) 0
7) $3x^2-6x-3$
8) $-8x^3+38x^2+6x$
9) $14x^2-21x+8$

Exercise 7

1) $\delta = \frac{-5}{6}$ en $\delta = 0$
2) $x = \frac{-5}{2}$ en $x = \frac{5}{2}$
3) $t = \frac{-3}{2}$ en $t = 2$
4) $x = -1$ en $x = \frac{-1}{4}$
5) $t = -2$
6) $t = -3$ en $t = 2$

Exercise 8 $V(x,y) = -3(a+7b)(x-2y)$

Exercise 9 $K(x) = 9\left(x - \frac{1}{3}\right)^2$

Exercise 10

1)	0	3)	2	5)	$-\frac{3}{2}$
2)	4	4)	$\frac{4}{3}$	6)	-4

2. Linear Systems

Exercise 11

$$\begin{cases} x &= 1 \\ y &= 0 \\ z &= 1 \\ v &= 2 \end{cases}$$

Exercise 12

$$\begin{cases} x_1 &= \frac{13}{2} \\ x_2 &= \frac{-3}{4} \\ x_3 &= -1 \end{cases}$$

Exercise 13 3 m by 1 m.

Exercise 14 123 and 87

Exercise 15 The father is 35 year and his son makes 11 year.

Exercise 16 Adison scores 260 points and Valence 210.

Exercise 17 In 2 hours 7 minutes 30 seconds the fastest robot produces 16 250 motherboards, while the slowest reaches 12 750.

Exercise 18 11 components of type I, 13 components of type II and 21 components of type III.

Exercise 19 The circumference is 60 cm.

Exercise 20 The length of the arm measures 41.41m.

Exercise 21 The aeroplane flies at a speed of 583.33 kms per hour and the wind-speed is 83.33 kms per hour.

Exercise 22 The price of one beer is 1.5 EUR.

Exercise 23 The parcel's dimensions are 3 cm by 4.5 cm by 2.5 cm.

3. Trigonometry

Exercise 24

1) $\beta = 42^\circ$
$a = 29 \tan 48^\circ \approx 32.21$
$c = \sqrt{29^2 + (29 \tan 48^\circ)^2} \approx 43.34$

2) $b = \sqrt{10^2 + 12^2 - 2 \cdot 12 \cdot 10 \cdot \cos 65^\circ} \approx 11.94$
$\gamma = \arcsin \frac{12 \sin 65^\circ}{b} \approx 65.62^\circ$
$\alpha = 180^\circ - 65^\circ - \gamma \approx 49.38$

Exercise 25 $\alpha = \arctan \frac{8}{20} - \arctan \frac{5}{20} \approx 7.77^\circ$

Exercise 26 $158.11m$

Exercise 27 $\frac{700 \ \sin 78^\circ}{\sin 69^\circ} \approx 733.42\ km$

Exercise 28 $\frac{250}{\sin 23^\circ}(\sin 80^\circ + \sin 77^\circ) \approx 1253.53\ km$

Exercise 29 $45\sqrt{3}$ metres

Exercise 30

1) $\alpha = 60^\circ$ en $\alpha = 120^\circ$
2) $\alpha = 45^\circ$ en $\alpha = 225^\circ$
3) $\alpha = 67,5^\circ$ en $\alpha = 112,5^\circ$
4) $\alpha = 240^\circ$ en $\alpha = 240^\circ$

Exercise 31

1) $x = 8.82\ m$, and $y = 11.46\ m$

2) $h = 13.19\ m$

4. Functions

Exercise 32

Setting the first slope $m = \tan\alpha$ indicates the second slope as $n = \tan(\alpha - \frac{\pi}{2})$. Therefore we deduct trigonometrically $n = -\tan(\frac{\pi}{2} - \alpha) = -\cot\alpha = -\frac{1}{\tan\alpha} = -\frac{1}{m}$.

Exercise 33

1) $a > 0, b > 0$ and $c < 0$
2) $a < 0, b < 0$ and $c < 0$
3) $a > 0, b > 0$ and $c > 0$
4) $a < 0, b > 0$ and $c > 0$

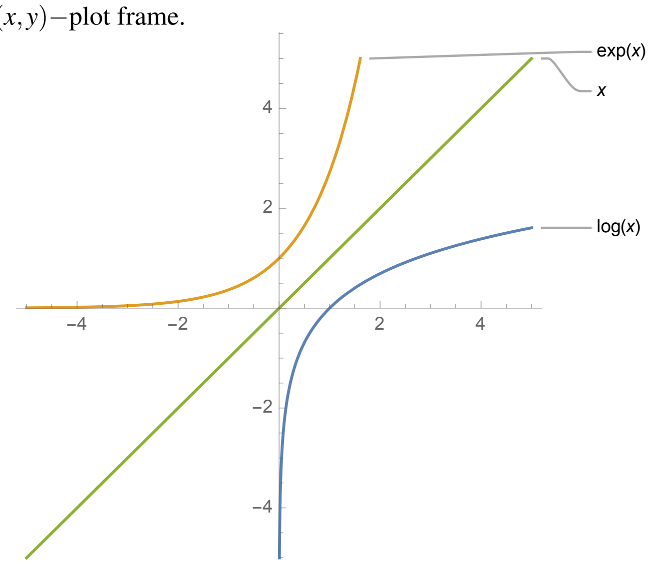
Exercise 34 $y = \frac{2}{3}x + 20$ and $y = \frac{-3}{2}x + 85$

Exercise 35 The car collides in the point $(1, 1)$.

Exercise 36 $(-1, -36)$ and $(4, 24)$

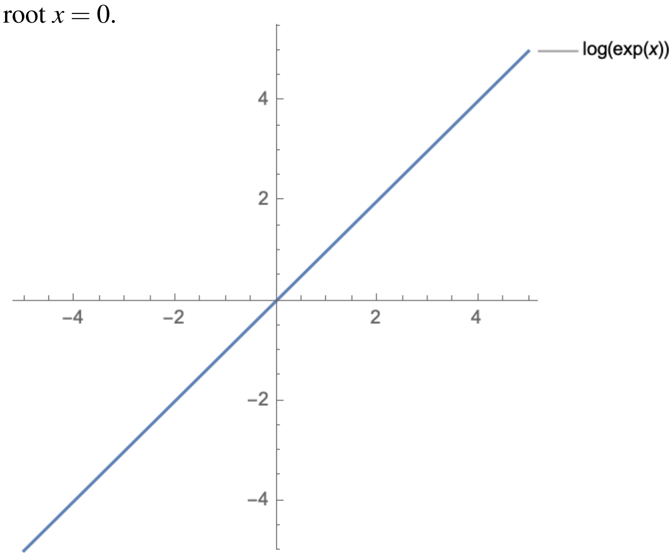
Exercise 37

- ▷ Inverting $y = \log_e(x)$ by solving $x = \log_e(y)$ for y yields $y = e^x = \exp(x)$.
- ▷ The graphs lie symmetric about the main diagonal $y = x$ in their orthonormal $(x, y)-$plot frame.

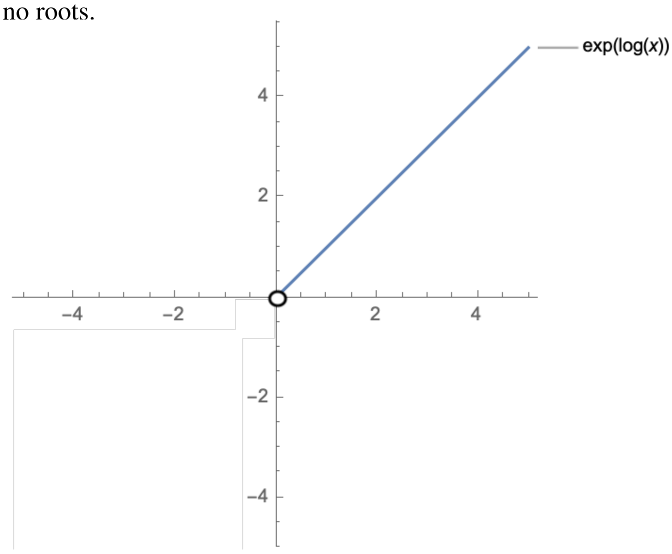


Exercise 38

- Composite function $f(x) = \log_e(\exp(x))$ has domain $\mathbb{R}$, range $\mathbb{R}$ and single root $x = 0$.



- Composite function $f(x) = \exp(\log_e(x))$ has domain $\mathbb{R}^+$, range $\mathbb{R}^+ \backslash \{0\}$ and no roots.



Exercise 39
One sheet of newspaper thickness is 0.08 mm.

Exercise 40

1) $2\sin\left(x - \frac{\pi}{2}\right) + 3$
2) $0.5\sin\left(2\left(x - \frac{\pi}{2}\right)\right) - 1$

Exercise 41
Approximatively $\frac{1}{9!} \approx 3 \times 10^{-6}$

5. The Golden Section

Exercise 42

If $|AB| = 1$ then is $|AC| = \frac{1}{2}$ and $|CB| = \frac{\sqrt{5}}{2}$. Given $|CB| = |CD| = |CA| + |AD|$ we conclude that $|AS| = |AD| = \frac{\sqrt{5}-1}{2} \overset{\text{see Exercise 42}}{=} \frac{1}{\Phi}$.

Exercise 43

If $|AB| = 1$ then is $|AM| = \frac{1}{2}$ and $|AT| = 1$. Calculating $|MT|$ yields $|MT| = \frac{\sqrt{5}}{2}$. Given $|MS| = |MT|$ we conclude $|AS| = |AM| + |MS| = \frac{1+\sqrt{5}}{2} = \Phi$.

Exercise 44

$$\begin{aligned}
\Phi^2 + \frac{1}{\Phi^2} &= \left(\frac{\sqrt{5}+1}{2}\right)^2 + \left(\frac{2}{\sqrt{5}+1}\right)^2 \\
&= \frac{3+\sqrt{5}}{2} + \frac{2}{3+\sqrt{5}} \\
&= \frac{(3+\sqrt{5})^2 + 2^2}{2(3+\sqrt{5})} \\
&= \frac{6(3+\sqrt{5})}{2(3+\sqrt{5})} \\
&= 3
\end{aligned}$$

Exercise 45

$$\begin{aligned}
\frac{1}{\Phi} &= \frac{2}{1+\sqrt{5}} \frac{1-\sqrt{5}}{1-\sqrt{5}} \\
&= \frac{2(1-\sqrt{5}}{-4} \\
&= \frac{\sqrt{5}-1}{2}
\end{aligned}$$

$$\begin{aligned}\Phi - 1 &= \frac{\sqrt{5}+1}{2} - 1 \\ &= \frac{\sqrt{5}-1}{2}\end{aligned}$$

Exercise 46

$$\begin{aligned}\Phi + \frac{1}{\Phi} &= \frac{1+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}} \\ &= \frac{(1+\sqrt{5})^2+2^2}{2(1+\sqrt{5}} \\ &= \frac{10+2\sqrt{5}}{2(1+\sqrt{5})} \\ &= \frac{2\sqrt{5}(\sqrt{5}+1)}{2(1+\sqrt{5})} \\ &= \sqrt{5}\end{aligned}$$

Exercise 47

- for $x_1 = -1$: $(-1)^3 + 2(-1)^2 - 1 = 0$
- for $x_2 = -\Phi$: $\left(-\left(\frac{1+\sqrt{5}}{2}\right)\right)^3 + 2\left(-\left(\frac{1+\sqrt{5}}{2}\right)\right)^2 - 1 = -\frac{16+8\sqrt{5}}{8} + \frac{6+2\sqrt{5}}{2} - 1 = 0$
- for $x_3 = -\Phi'$: $\left(-\left(\frac{1-\sqrt{5}}{2}\right)\right)^3 + 2\left(-\left(\frac{1-\sqrt{5}}{2}\right)\right)^2 - 1 = -\frac{16-8\sqrt{5}}{8} + \frac{6-2\sqrt{5}}{2} - 1 = 0$

Exercise 48

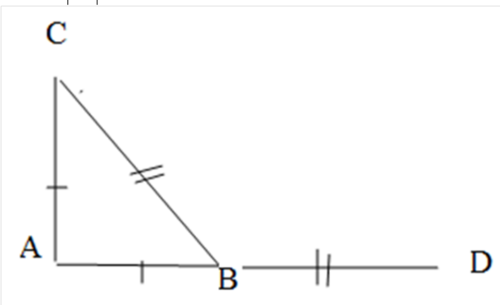
- $\Phi^2 = \Phi + 1$
- $\Phi^3 = \Phi\,\Phi^2 = \Phi\,(\Phi+1) = \Phi^2 + \Phi = 2\Phi + 1$
- $\Phi^4 = \Phi\,\Phi^3 = \Phi\,(2\Phi+1) = 2\Phi^2 + \Phi = 3\Phi + 2$
- $\Phi^5 = \Phi\,\Phi^4 = \Phi\,(3\Phi+2) = 3\Phi^2 + 2\Phi = 5\Phi + 3$

Exercise 49

month	0	1	2	3	4	5	6	7	8	9	10	11	12
couples of rabbits	1	1	2	3	5	8	13	21	34	55	89	144	233

Exercise 50

1) We obtain the silver section by extending one of the sides of an isosceles right triangle by its hypotenuse. Hence the line segment $|AD|$ relates to $|AB|$ as $\delta = \frac{|AD|}{|AB|} = \frac{(1+\sqrt{2})|AB|}{|AB|} = 1+\sqrt{2}$.



2) Solving the geometric ratio $\frac{x}{1} = \frac{1}{x-2}$ mathematically as the quadratic equation $x^2-2x-1=0$ for x, yields a discriminant $D=8$ and the two roots $x_1 = 1+\sqrt{2}$ and $x_2 = 1-\sqrt{2}$.

3) By dividing the definition of the sequence of Pell by p_n and subsequently substituting its ratio by $D = \frac{p_n}{p_{n-1}}$, we rewrite the number sequence of Pell in its limit as $D = 2+\frac{1}{D}$. Therefore, the standardized equation $D^2-2D-1=0$ describes the ratio D for incrementing indices without bound. It yields the positive root $D_1 = 1+\sqrt{2} = \delta$ as meaningful answer to the above.

6. Coordinate systems

Exercise 51

1) $(0,1)_{cc}$
2) $\left(\frac{3\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2}\right)_{cc}$
3) $(0,-4)_{cc}$
4) $\left(-\sqrt{2}, -\sqrt{2}\right)_{cc}$

Exercise 52

1) $\left(2\sqrt{2}, \frac{\pi}{4}\right)_{pc}$
2) $\left(1, \frac{3\pi}{2}\right)_{pc}$
3) $\left(1, \frac{\pi}{3}\right)_{pc}$
4) $\left(2, \frac{7\pi}{4}\right)_{pc}$

Exercise 53

$$\left(\frac{-10}{4}, \frac{5\sqrt{5}}{2}\right)_{cc} = \left(\sqrt{\frac{75}{2}}, \arctan(-\sqrt{5})\right)_{pc}$$

after rotation becomes

$$\left(\sqrt{\frac{75}{2}}, \arctan(-\sqrt{5}) + \frac{\pi}{2}\right)_{pc} = \left(\frac{-5\sqrt{5}}{2}, \frac{-10}{4}\right)_{cc}$$

Exercise 54

1) $\tan\theta = 3$

2) $r = \frac{3}{2\cos\theta+\sin\theta}$

3) $r = -2\cos\theta$

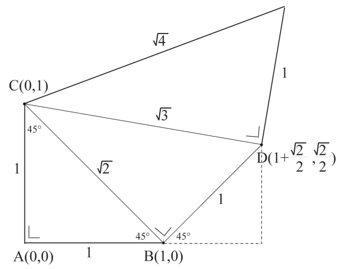
Exercise 55

1) $y = \pm\sqrt{2x+1}$

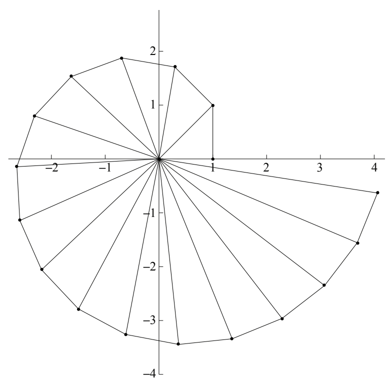
2) $\sqrt{x^2+y^2} = \text{atan2}(y,x)$

Exercise 56

The vertex D has coordinates $\left(1+\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The growing radial coordinates are the successive square roots of the natural numbers: $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots$



The spiral of Theodore of Cyrene, also known as the 'square root spiral', is the oldest mathematical spiral. For instance starting from the vertices $A(0,0), B(1,0)$ and $C(1,1)$ it yields in 16 steps the included figure.



Exercise 57

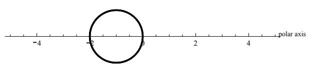
Applying the Pythagorean Identity yields:

$$\begin{cases} \dfrac{x\sqrt{t}}{a} = \cos t \\ \dfrac{y\sqrt{t}}{a} = \sin t \end{cases} \Longrightarrow \begin{cases} \left(\dfrac{x\sqrt{t}}{a}\right)^2 = (\cos t)^2 \\ \left(\dfrac{y\sqrt{t}}{a}\right)^2 = (\sin t)^2 \end{cases} \Longrightarrow \frac{t}{a^2}(x^2+y^2) = 1 \Longrightarrow \frac{t}{a^2}r^2 = 1,$$

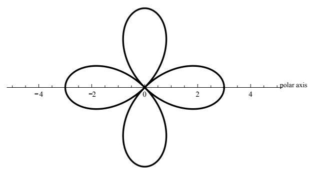
and solving it for r gives $r = \dfrac{a}{\sqrt{t}}$ with its polar angle $t \in \mathbb{R}^+ \backslash \{0\}$.

Exercise 58

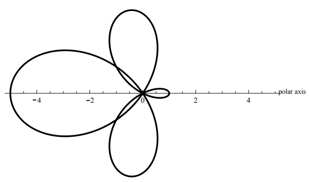
1) $a = 2, b = 0$



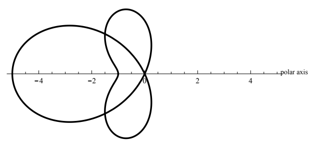
2) $a = 0, b = 3$



3) $a = 2, b = 3$



4) $a = 3, b = 2$



Exercise 59

1) $m = 0$

2) $a = 2, b = 4, m = 4, \; n_1 = 2, n_2 = 2, n_3 = 2$

7. Vectors

Exercise 60

$\|\vec{F}\| = \sqrt{(10-20\sqrt{2}+60\sqrt{3})^2+(60-20\sqrt{2}+90\sqrt{3})^2} = 206.22$ and
$\alpha = \arctan\frac{60-20\sqrt{2}+90\sqrt{3}}{10-20\sqrt{2}+60\sqrt{3}} = 1.14 \text{ rad}$

Exercise 61

$$\vec{v}+\vec{w} = \begin{pmatrix} 800\cos 240^\circ + 90\cos 105^\circ \\ 800\sin 240^\circ + 90\sin 105^\circ \end{pmatrix} = \begin{pmatrix} -423.294 \\ -605.887 \end{pmatrix}$$

ground speed:

$$\sqrt{(800\cos 240^\circ + 90\cos 105^\circ)^2 + (800\sin 240^\circ + 90\sin 105^\circ)^2} = 739.11 \text{ kms per hour}$$

direction of ground velocity:

$$\begin{aligned} \arctan\frac{800\sin 240^\circ + 90\sin 105^\circ}{800\sin 240^\circ + 90\sin 105^\circ} &= 0.96+\pi = 4.10 \text{ rad} \\ &= 235.06^\circ \text{ (referred to the positive } x\text{-axis)} \\ &= 235.06^\circ \text{ (referred to the North)} \end{aligned}$$

Exercise 62

1) $\begin{pmatrix} 99 \\ 0 \\ -128 \end{pmatrix}$

2) $\begin{pmatrix} -60 \\ -326 \\ 256 \end{pmatrix}$

3) -428

4) $\begin{pmatrix} 112 \\ -22 \\ -92 \end{pmatrix}$

5) $\begin{pmatrix} 0 \\ -28 \\ 0 \end{pmatrix}$

6) $\begin{pmatrix} 104 \\ 88 \\ 122 \end{pmatrix}$

Exercise 63

The angle between the vector $\vec{f}$ and the vector from the camera's point to the object, is less than 90°, hence the object is captured.

Exercise 64

1) $|HB| = 0.3$ and $A(1.5,0,0), B(1.8,0,-0.4), C(1.8,1,-0.4), D(1.5,1,0), F(1.5,0.5,0)$

2) $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} 0.4 \\ 0 \\ 0.3 \end{pmatrix}$

3) $G(1.74, 0.5, 0.18)$

Exercise 65 $\sqrt{2390}$

Exercise 66 $70.89°$

8. Parameters

Exercise 67

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Exercise 68

1) $\begin{cases} x = 1 - 4\lambda \\ y = -4\lambda \\ z = 4 - 8\lambda \end{cases}$

2) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

Exercise 69

1) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ or $\begin{cases} x = 4 - \lambda \\ y = 2 + 4\lambda \\ z = 8 + 3\lambda \end{cases}$ or $\dfrac{4-x}{1} = \dfrac{y-2}{4} = \dfrac{z-8}{3}$

2) $\begin{cases} x = 5 + 2\mu \\ y = 8 + 2\mu \\ z = 21 + 10\mu \end{cases}$ and this is equivalent to $\dfrac{x-5}{2} = \dfrac{y-8}{2} = \dfrac{z-21}{10}$

3) $S(3,6,11)$

Exercise 70

1) $x-2=-y+2$ and $z=3$

2) $\begin{cases} x=5+\lambda \\ y=5+\lambda \\ z=5+\mu \end{cases}$

$\begin{cases} 8=5+\lambda \\ 8=5+\lambda \\ 4=5+\mu \end{cases}$ has a solution for $\lambda=1$ and $\mu=-1$, hence $P \in v_C$

Exercise 71

1) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix}$

2) $\vec{n} = \begin{pmatrix} -8 \\ 23 \\ -56 \end{pmatrix}$

Exercise 72

1) $\begin{cases} x=3+\lambda+4\mu \\ y=6+2\lambda+2\mu \\ z=2+3\lambda+\mu \end{cases}$

2) $z=\frac{-4}{6}x+\frac{11}{6}y-7$

Exercise 73

1) $\begin{cases} x=\lambda+5\mu \\ y=2\lambda-\mu \\ z=2\lambda+\mu \end{cases}$

2) $z=\frac{4}{11}x+\frac{9}{11}y$

3) $\vec{n} = \begin{pmatrix} 4 \\ 9 \\ -11 \end{pmatrix}$

Exercise 74

The lines are not parallel because of their different direction vectors. And since the system $\begin{cases} -8+5\lambda = 4+9\mu \\ 2+2\lambda = -1+\mu \\ -4+3\lambda = 2+6\mu \end{cases}$ has no solution, the lines are not intersecting as well. Hence both lines are skew lines.

Exercise 75 $S\left(\frac{212}{97}, \frac{-79}{97}, \frac{-61}{97}\right)$

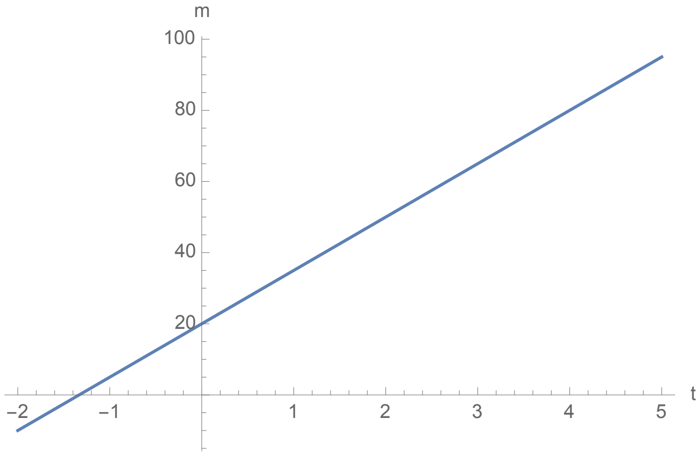
Exercise 76
The intersection of the three planes is the line by parameter equation

$$\begin{cases} x = \frac{2}{3} + \lambda \frac{1}{3} \\ y = \frac{1}{3} + \lambda \frac{2}{3} \end{cases}$$

Exercise 77
The intersection consists of the two points $S_1\left(1-4\sqrt{2}, 2(1+\sqrt{2}), 3(1-\sqrt{2})\right)$ and $S_2\left(1+4\sqrt{2}, 2(1-\sqrt{2}), 3(1+\sqrt{2})\right)$

9. Kinematics

Exercise 78 For the real location function $s(t) = 20 + 15t$ we conclude



its domain $s = \mathbb{R}$,

its range $s = \mathbb{R}$,

its root lying at $t_0 = -1.333$

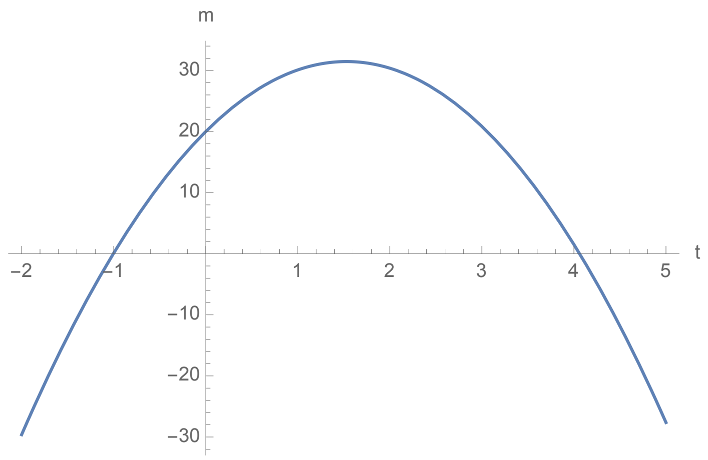
Exercise 79

For the real location function $s(t) = 20 + 15t + \frac{1}{2}(-9.81)t^2$ we conclude

its domain $s = \mathbb{R}$,

its range $s =]-\infty, 31.47]$,

its roots lying at $t_1 = -1.004, t_2 = 4.062$



Exercise 80 For the velocity function $v(t) = 8.2 + (-9.81)t$ we evaluate

1) $v(0.5) = 3.295 \ \frac{m}{s}$,
2) $v(1) = -1.610 \ \frac{m}{s}$.

Exercise 81

$$y(t) = 3\sin\left(\frac{2\pi}{3}t + \arcsin\left(\frac{1}{3}\right)\right)$$

Exercise 82

1) $\|\vec{\omega}\| = \omega = 66.67 \ \frac{\text{rad}}{s}$ with $\vec{\omega}$ perpendicular to the wheel(disk),
2) 10.61 (wheel)turns per second,
3) 1000 rad consumed,
4) $\|\vec{a}_c\| = a_c = 1334 \ \frac{m}{s^2}$.

Exercise 83

1) $\vec{a}$ is in T directed downhill,
2) $\vec{a}_c$ is in U directed (centripetally) upwards,

3) $\vec{g}$ is in J directed downwards.

Exercise 84

1) $v = 2.7\ \frac{m}{s}$ and $a_t = 0.27\ \frac{m}{s^2}$,
2) $a_c = 2.700\ \frac{m}{s^2}$ and hence $\|\vec{a}\| = 2.713\ \frac{m}{s^2}$.

Exercise 85 Centering the polar coordinate system (r, θ) at the Wheel's axis:

$\vec{s} = -r\vec{e}_n + 0\vec{e}_t$
$\vec{v} = 0\vec{e}_n + r\omega\vec{e}_t$
$\vec{a} = r\omega^2\vec{e}_n + 0\vec{e}_t$

$\vec{s} = r\vec{e}_r + 0\vec{e}_\theta$
$\vec{v} = 0\vec{e}_r + r\omega\vec{e}_\theta$
$\vec{a} = -r\omega^2\vec{e}_r + 0\vec{e}_\theta$

Exercise 86

1) Both masses land simultaneously (Independence of Motion Principle).
2) This took them 3.375 seconds.
3) As the first mass did not, the second traveled 33.75 meter horizontally.

Exercise 87

The rider and his motorcycle reach a maximum height of $y_T = 20.16\ m$ before they touch down after flying $x = 202.1\ m$ horizontally.

10. Collision detection

Exercise 88

1) $B((1,3,5),4)$
2) $B((3,6,-3),2)$
3) is not a circle

Exercise 89

1) $C\left((1,2), \sqrt{5}\right)$
2) The center of the circle is the intersection point of the perpendicular bisectors (see page 46) on line segments $[AB]$ and $[BC]$, given $A(0,0), B(2,0)$ and $C(0,4)$.

Exercise 90 $(x-20)^2+(y-50)^2=50^2$

Exercise 91 no collision

Exercise 92 Both circumscribed circles are tangent in the point $T(1,-1)$.

Exercise 93 $d=\dfrac{\sqrt{3}}{\sqrt{14}}\approx 0.46$

Exercise 94 $d(S,v_A)=0.07<5$, hence we have a collision

Exercise 95 For any $\vec{v}\neq\vec{o}$ we satisfy $(\vec{v}\cdot\vec{p}-\vec{v}\cdot\vec{q})(\vec{v}\cdot\vec{r}-\vec{v}\cdot\vec{q})\leqslant 0$. Hence the point Q lies in between the points P and R.

Exercise 96

1) $d(V,v_O)=11$
2) $d(V,v_O)=0$ and the point V lies in the polygon with vertices P,Q and O, hence it is a successful landing.

Exercise 97 $d(S,v_A)=0$ and the point S lies in the polygon with vertices A,B and C, hence the snooker ball lies in the triangle ABC filled with numbered balls.

Exercise 98 Collision occurs between the second and the third frame according to:

1) $d(P,v_A)=\frac{9}{5}$
2) $d(P,v_A)=\frac{4}{5}$
3) $d(P,v_A)=\frac{-1}{5}$
4) $d(P,v_A)=\frac{-6}{5}$

Exercise 99 In the third frame the squared distance is less than 225, hence we have a collision. In other words, collision occurs between the second and the third frame.

11. Matrices

Exercise 100

1) $\begin{pmatrix} 9 & 20 & 3 & 1 \\ 20 & 280 & 50 & 70 \\ 30 & 40 & 4 & 1 \end{pmatrix}$

2) $\begin{pmatrix} -7 & -10 & 6 & -3 \\ -10 & -440 & 50 & 40 \\ -40 & 30 & -2 & -3 \end{pmatrix}$

3) $\begin{pmatrix} 5 & 10 & 20 \\ 10 & 200 & 10 \\ 0 & 10 & 2 \\ 1 & 20 & 1 \end{pmatrix}$

4) $\begin{pmatrix} 4 & 10 & 10 \\ 10 & 80 & 30 \\ 3 & 40 & 2 \\ 0 & 50 & 0 \end{pmatrix}$

5) $\begin{pmatrix} -1 & 0 & -10 \\ 0 & -120 & 20 \\ 3 & 30 & 0 \\ -1 & 30 & -1 \end{pmatrix}$

6) $\begin{pmatrix} -1 & 0 & -10 \\ 0 & -120 & 20 \\ 3 & 30 & 0 \\ -1 & 30 & -1 \end{pmatrix}$

Exercise 101

1) $\begin{pmatrix} 8 & 8 \\ 1 & 4 \end{pmatrix}$

2) $\begin{pmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 9 \end{pmatrix}$

3) $\begin{pmatrix} 1 & -3 & -3 & -3 \\ 0 & 3 & 1 & 4 \\ 2 & 9 & -1 & 14 \end{pmatrix}$

4) $C \cdot B$ does not exist

5) $\begin{pmatrix} 8 & 24 & -8 & 40 \\ 1 & 12 & 2 & 17 \end{pmatrix}$

6) $\begin{pmatrix} 8 & 24 & -8 & 40 \\ 1 & 12 & 2 & 17 \end{pmatrix}$

7) $\begin{pmatrix} 8 & 1 \\ 8 & 4 \end{pmatrix}$

8) $\begin{pmatrix} 8 & 1 \\ 8 & 4 \end{pmatrix}$

Exercise 102

1) $\begin{pmatrix} 70 & 20 \\ 30 & 10 \end{pmatrix}$

2) $\begin{pmatrix} \frac{1}{5} & \frac{-1}{5} \\ \frac{-3}{10} & \frac{4}{5} \end{pmatrix}$

5) D^{-1} does not exist

6) $\begin{pmatrix} 1 & \frac{-7}{5} \\ \frac{-5}{2} & \frac{18}{5} \end{pmatrix}$

3) $\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$

4) $\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$

7) $\begin{pmatrix} \frac{9}{5} & \frac{-19}{5} \\ \frac{-13}{10} & \frac{14}{5} \end{pmatrix}$

8) $\begin{pmatrix} \frac{9}{5} & \frac{-19}{5} \\ \frac{-13}{10} & \frac{14}{5} \end{pmatrix}$

Exercise 103

$$\begin{pmatrix} -1 & 1 & -4 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} & \frac{7}{4} & -1 \\ \frac{1}{2} & \frac{-5}{4} & 1 \\ 0 & \frac{-3}{4} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-1}{2} \\ -2 \end{pmatrix}$$

Exercise 104

if $k = 1$ we have $\vec{f_1} = F\vec{f_0}$
if $k = 2$ we have $\vec{f_2} = F\vec{f_1} = F^2\vec{f_0}$
if $k = 3$ we have $\vec{f_3} = F\vec{f_2} = F^3\vec{f_0}$
$\vdots$
if we assume $\vec{f_k} = F^k\vec{f_0}$ given $k \in \mathbb{N}$
then for $k+1$ we get $\vec{f_{k+1}} = F\vec{f_k} = F^{k+1}\vec{f_k}$

Exercise 105

$$\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1-\sqrt{5}}{2\sqrt{5}}, \frac{1}{\sqrt{5}} \\ -\frac{-1-\sqrt{5}}{2\sqrt{5}}, -\frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = F^1$$

$$\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{6+2\sqrt{5}}{4} & 0 \\ 0 & \frac{6-2\sqrt{5}}{4} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1-\sqrt{5}}{2\sqrt{5}}, \frac{1}{\sqrt{5}} \\ -\frac{-1-\sqrt{5}}{2\sqrt{5}}, -\frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = F^2$$

Exercise 106

$$f_0 = \frac{\Phi^0 - \Phi'^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0$$
$$f_1 = \frac{\Phi^1 - \Phi'^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} = 1$$
$$f_2 = \frac{\Phi^2 - \Phi'^2}{\sqrt{5}} = \frac{\frac{6+2\sqrt{5}}{4} - \frac{6-2\sqrt{5}}{4}}{\sqrt{5}} = \frac{\frac{4\sqrt{5}}{4}}{\sqrt{5}} = 1$$
$$f_3 = \frac{\Phi^3 - \Phi'^3}{\sqrt{5}} = \frac{(2+\sqrt{5})-(2-\sqrt{5})}{\sqrt{5}} = \frac{2\sqrt{5}}{\sqrt{5}} = 2$$

Exercise 107

$$K \cdot K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$L \cdot L = \begin{pmatrix} 1-a & \frac{(1-a)a}{b} \\ b & a \end{pmatrix}$$

$$M \cdot M = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 108

$$N \cdot N = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P \cdot P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$R \cdot R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

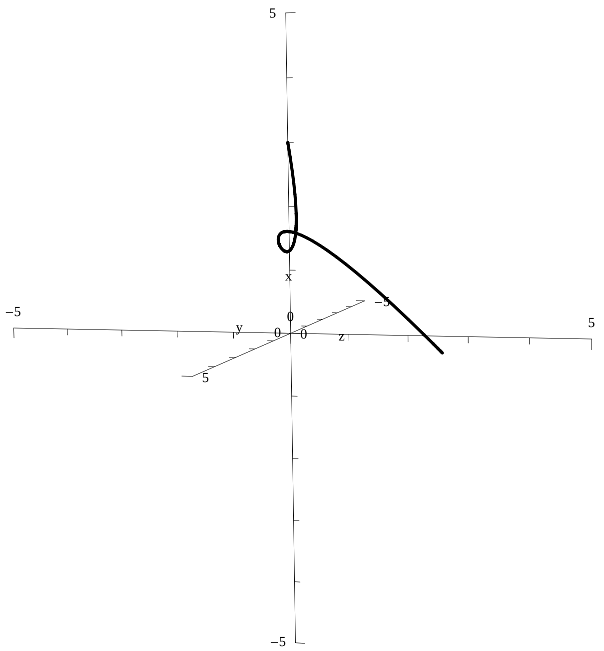
12. Bezier curves

Exercise 109

$$\begin{aligned}
\vec{b}_{021}(t) &= (1-t)^2\vec{p}_0 + 2(1-t)t\vec{p}_2 + t^2\vec{p}_1 \\
&= (1-t)^2\vec{p}_0 + 2(1-t)t\left(\frac{\vec{p}_0+\vec{p}_1}{2}\right) + t^2\vec{p}_1 \\
&= (1-t)^2\vec{p}_0 + (1-t)t\,(\vec{p}_0+\vec{p}_1) + t^2\vec{p}_1 \\
&= (1-2t+t^2)\vec{p}_0 + (t-t^2)\,(\vec{p}_0+\vec{p}_1) + t^2\vec{p}_1 \\
&= \vec{p}_0 - 2t\vec{p}_0 + t^2\vec{p}_0 + t\vec{p}_0 - t^2\vec{p}_0 + t\vec{p}_1 - t^2\vec{p}_1 + t^2\vec{p}_1 \\
&= (1-t)\vec{p}_0 + t\vec{p}_1 = \vec{b}_{01}(t)
\end{aligned}$$

Exercise 110

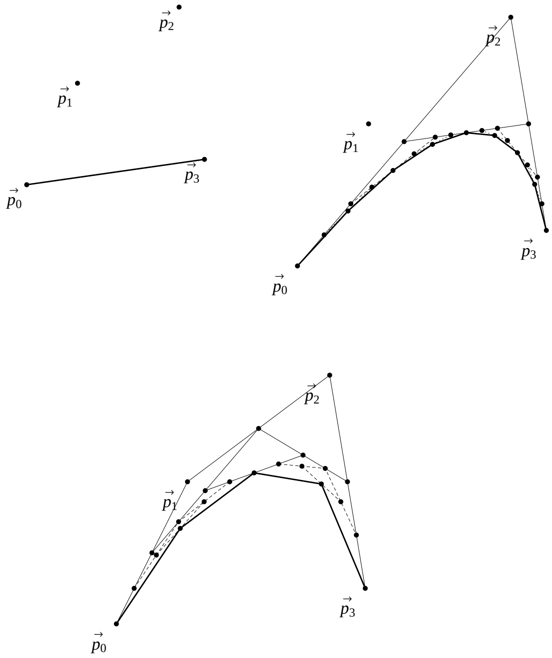
$$\vec{b}_{0123}(t) = \begin{cases} x(t) = 3t - 12t^2 + 12t^3 \\ y(t) = 3 - 12t + 27t^2 - 18t^3 \\ z(t) = 3t - 6t^2 + 5t^3 \end{cases}$$



Exercise 111

$$
\begin{aligned}
\text{The sum of the coefficients of } \vec{b}_{012}(t) &= (1-t)^2 + 2(1-t)t + t^2 \\
&= 1 - 2t + t^2 + 2t - 2t^2 + t^2 \\
&= 1, \\
\text{The sum of the coefficients of } \vec{b}_{0123}(t) &= (1-t)^3 + 3(1-t)^2 t + 3(1-t)t^2 + t^3 \\
&= 1 - 3t + 3t^2 - t^3 + 3(1 - 2t + t^2)t + 3(t^2 - t^3) + t^3 \\
&= 1 - 3t + 3t^2 - t^3 + 3t - 6t^2 + 3t^3 + 3t^2 - 3t^3 + t^3 \\
&= 1.
\end{aligned}
$$

Exercise 112

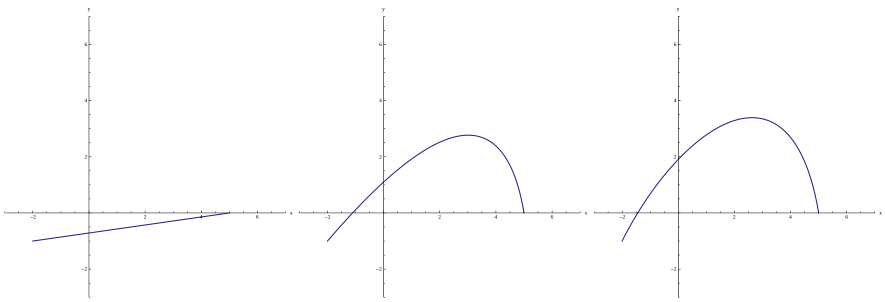


Exercise 113

1) $\vec{b}_{03}(t) = \begin{cases} x(t) = -2+7t \\ y(t) = -1+t \end{cases}$

2) $\vec{b}_{023}(t) = \begin{cases} x(t) = -2+12t-5t^2 \\ y(t) = -1+14t-13t^2 \end{cases}$

3) $\vec{b}_{0123}(t) = \begin{cases} x(t) = -2+6t+6t^2-5t^3 \\ y(t) = -1+12t-3t^2-8t^3 \end{cases}$

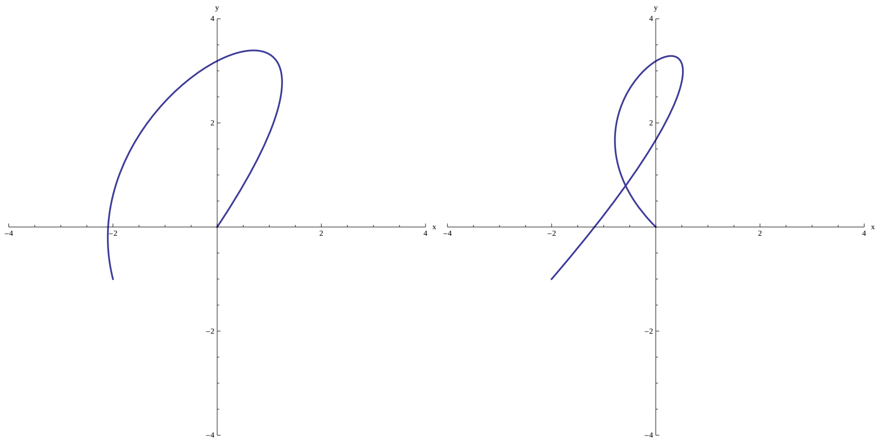


Exercise 114

1) $\vec{b}_{0123}(t) = \begin{cases} x(t) = -2 - 3t + 24t^2 - 19t^3 \\ y(t) = -1 + 12t - 3t^2 - 8t^3 \end{cases}$

2) $\vec{b}_{0123}(t) = \begin{cases} x(t) = -2 + 18t - 39t^2 + 23t^3 \\ y(t) = -1 + 21t - 30t^2 + 10t^3 \end{cases}$

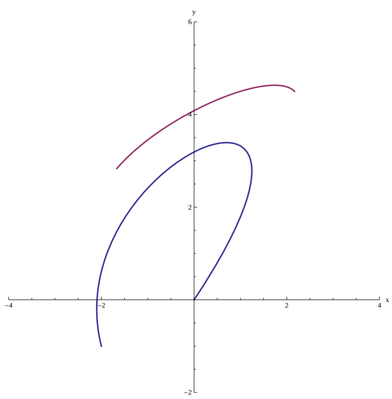
We notice this second plane cubic Bezier segment has a looped profile.



Exercise 115

$$\vec{b}_{0123}(t) = \begin{cases} x(t) = -2 - 3t + 24t^2 - 19t^3 \\ y(t) = -1 + 12t - 3t^2 - 8t^3 \end{cases}$$

$$\vec{s}_{0123}(t) = \begin{cases} x(t) = -\frac{5}{3} + 3t + 4t^2 - \frac{19}{6}t^3 \\ y(t) = \frac{17}{6} + \frac{7}{2}t - \frac{1}{2}t^2 - \frac{4}{3}t^3 \end{cases}$$



Exercise 116 Evaluating the initial and final parameter value yields $\vec{s}_{0123}(0) \neq \overrightarrow{p_0}$ and, after simplifying, also $\vec{s}_{0123}(1) \neq \overrightarrow{p_3}$.

Exercise 117

$$\vec{s}_{0123}(t) = \begin{cases} x(t) = \frac{1}{3} - t - 2t^2 + \frac{5}{3}t^3 \\ y(t) = \frac{7}{6} - \frac{3}{2}t + \frac{1}{2}t^2 \end{cases}$$

Exercise 118

1) $\vec{b}_{01}(t) = \begin{cases} x(t) = -1 + 2t \\ y(t) = 2t \end{cases}$

2) $\vec{b}_{012}(t) = \begin{cases} x(t) = -1 + 4t - 5t^2 \\ y(t) = 4t - 3t^2 \end{cases}$

3) $\vec{b}_{0123}(t) = \begin{cases} x(t) = -1 + 6t - 15t^2 + 10t^3 \\ y(t) = 6t - 9t^2 + 6t^3 \end{cases}$

4) $y = x + 1$

5) We obtain the parameter values t of the occasional intersection points by substituting the parametric equation of $\vec{b}_{0123}(t)$ into the cartesian equation of $\vec{b}_{01}(t)$.

$$y = x + 1 \;\Rightarrow\; 6t - 9t^2 + 6t^3 = (-1 + 6t - 15t^2 + 10t^3) + 1 \;\Rightarrow\; t = 0 \vee t = 0 \vee t = \frac{3}{2}$$

The mathematically found parameter value $t = \frac{3}{2}$ is for this occasion meaningless because of the constraint $t \in [0,1] \subset \mathbb{R}$. Substituting the remaining parameter value $t = 0$ (of multiplicity 2) into the linear Bezier segment $\vec{b}_{01}(t)$ yields the intersection point $\vec{b}_{01}(0) = (-1, 0)$ (of multiplicity 2).

13. Transformations

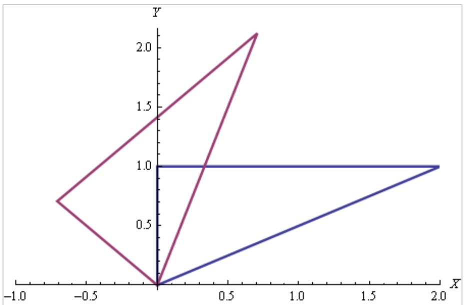
Exercise 119

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0,5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -50 & -20 \\ 30 & 100 & 0 \\ -100 & -20 & -300 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -100 & -40 \\ 30 & 100 & 0 \\ -50 & -10 & -150 \\ 1 & 1 & 1 \end{pmatrix}$$

Exercise 120

$$\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{3\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 1 & 1 \end{pmatrix}$$



Exercise 121

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 3-\sqrt{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1-2\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 3-\sqrt{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1-2\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 8 & 7 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{\sqrt{2}}+3-\sqrt{2} & \frac{6}{\sqrt{2}}+3-\sqrt{2} & \frac{4}{\sqrt{2}}+3-\sqrt{2} \\ \frac{7}{\sqrt{2}}+1-2\sqrt{2} & \frac{10}{\sqrt{2}}+1-2\sqrt{2} & \frac{10}{\sqrt{2}}+1-2\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}$$

Exercise 122

1) The plane basic shearing conserves area: shearing transforms rectangles into parallelograms, which have the same formula for area.

2) The inverse basic shearing takes the opposite angle: straightforward matrix calculations prove

$$S_{\sigma_x} \cdot S_{-\sigma_x} = I_3 \qquad \text{and} \qquad S_{\sigma_y} \cdot S_{-\sigma_y} = I_3.$$

3) Straightforward matrix calculations prove

$$S_{\sigma_x,\sigma_y} \cdot S_{-\sigma_x,-\sigma_y} \neq I_3.$$

4) Straightforward matrix calculations also prove

$$S_{\sigma_x,\sigma_y} \neq S_{\sigma_x} \cdot S_{\sigma_y} \qquad \text{and} \qquad S_{\sigma_x,\sigma_y} \neq S_{\sigma_y} \cdot S_{\sigma_x}.$$

Exercise 123

$$\begin{pmatrix} \cos\frac{\pi}{4} & 0 & \sin\frac{\pi}{4} & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ 0 & \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{-3-\sqrt{3}}{2\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{2} & \frac{-1}{2} & \frac{-\sqrt{3}+1}{2} \\ \frac{-1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

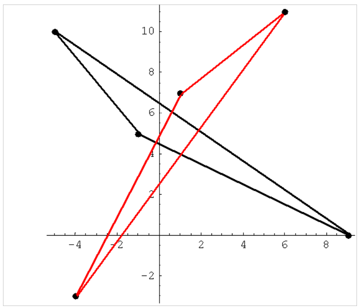
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{-3-\sqrt{3}}{2\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{2} & \frac{-1}{2} & \frac{-\sqrt{3}+1}{2} \\ \frac{-1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{-3-\sqrt{3}}{2} & \frac{-1-\sqrt{3}}{2\sqrt{2}} & \frac{-2-\sqrt{3}}{2\sqrt{2}} & \frac{-3}{2\sqrt{2}} \\ \frac{-\sqrt{3}+1}{2} & \frac{-\sqrt{3}+1}{2} & \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}+1}{2} & \frac{-\sqrt{3}-1}{2} & \frac{2-\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Exercise 124

▷ The centroid is $Z = (1,5)$.

▷ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -4 \\ -1 & 0 & 6 \\ 0 & 0 & 1 \end{pmatrix}$

▷ $\begin{pmatrix} 0 & 1 & -4 \\ -1 & 0 & 6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -5 & -1 & 9 \\ 10 & 5 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 4 \\ 11 & 7 & -3 \\ 1 & 1 & 1 \end{pmatrix}$



Exercise 125

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 4 & 0 & -6 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 4 & 0 & -6 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 & 5 & 2 & 2 & 5 & 5 & 2 \\ 2 & 1 & 1 & 2 & 5 & 4 & 1 & 5 \\ 1 & 2 & -1 & -1 & 1 & 2 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 8 & 8 & 2 & 2 & 8 & 8 & 2 \\ 2 & -2 & -2 & 2 & 14 & 10 & -2 & 14 \\ 1 & 4 & -5 & -5 & 1 & 4 & 10 & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Exercise 126

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 5 & 4 & 2 & 1 \\ 2 & 2 & 3 & 4 & 4 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 3 & 2 & 2 & 3 \\ 2 & 4 & 5 & 4 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Exercise 127

Firstly, we translate over a distance 1 upwards. Secondly, we rotate around the origin O over an angle $-\theta$, given $\theta = \arctan(2)$. Essentially, we reflect over the x-axis. Nextly, we inversely rotate over the angle θ, given $\theta = \arctan(2)$. Finally, we inversely translate over the distance 1 downwards.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{-3}{5} & \frac{4}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} & \frac{-2}{5} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-3}{5} & \frac{4}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} & \frac{-2}{5} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 5 & 6 & \frac{9}{2} \\ 1 & 2 & 4 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -0.8 & -0.6 & 0.4 & 0.5 \\ 3.4 & 4.8 & 6.8 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Exercise 128 Applying row reductions to calculate the inverse matrix operators for each, yields respectively

- ▷ the inverse translation matrix $T^{-1}_{\overrightarrow{AB}} = T_{\overrightarrow{BA}}$
- ▷ the inverse basic scale operator

$$S_O^{-1} = \begin{pmatrix} \frac{1}{s_x} & 0 & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 & 0 \\ 0 & 0 & \frac{1}{s_z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

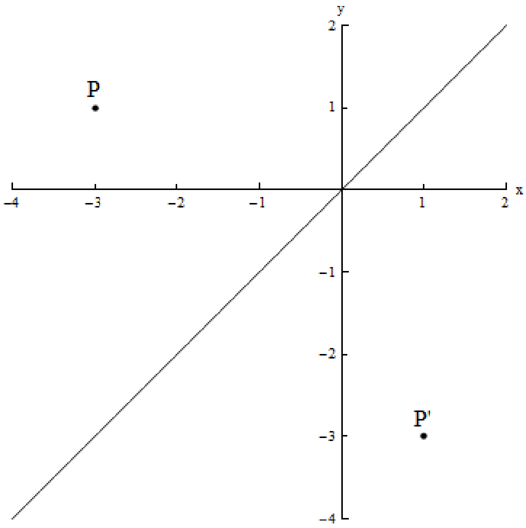
- ▷ the inverse basic rotation operator $R^{-1}_{O,\theta} = R_{O,-\theta}$

Exercise 129

We rotate around the origin O over the angle $-\frac{\pi}{4}$. Nextly, we reflect over the x-axis. Finally, we inversely rotate over the angle $\frac{\pi}{4}$.

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$



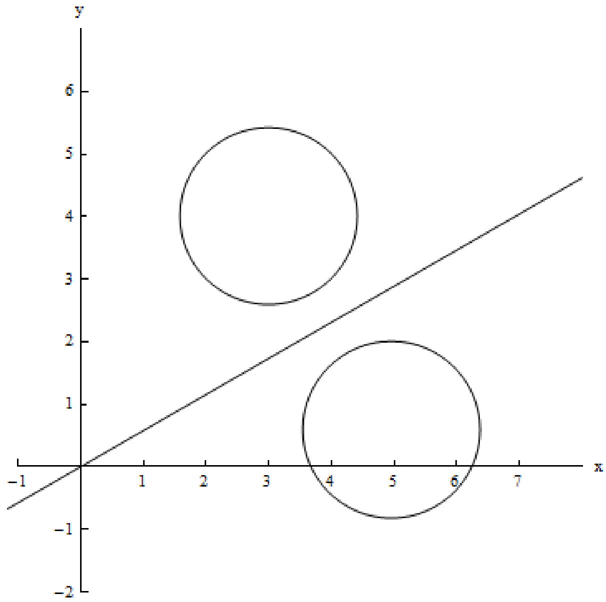
Exercise 130

We rotate around the origin O over the angle $-30°$. Nextly, we reflect over the x-axis. Finally, we inversely rotate over the angle $30°$.

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4.96 \\ 0.60 \\ 1 \end{pmatrix}$$

The radius of the circles equals $d(P,M) = \sqrt{2}$.

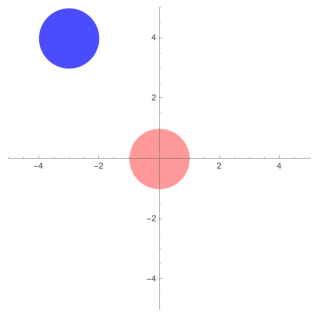


14. Transformation analysis

Exercise 131 We translate the origin O by the displacement $\vec{t} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ by applying the matrix operator $\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$ on the left hand side:

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$$

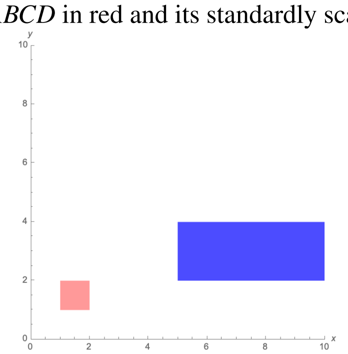
We draw unit disks (to better envision the circles) around the origin O in red, and its translation image O' in blue.



Exercise 132 We scale the square $ABCD$ by applying the standard scaling matrix operator $\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on the left hand side:

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 10 & 10 \\ 2 & 4 & 4 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

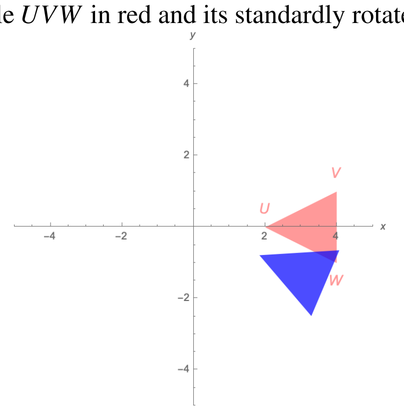
We draw the square $ABCD$ in red and its standardly scaled image $A'B'C'D'$ in blue.



Exercise 133 We rotate the triangle UVW by applying the standard rotation matrix operator $\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on the left hand side:

$$\begin{pmatrix} \cos(-23^\circ) & -\sin(-23^\circ) & 0 \\ \sin(-23^\circ) & \cos(-23^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1.84 & 4.07 & 3.29 \\ -0.78 & -0.64 & -2.48 \\ 1 & 1 & 1 \end{pmatrix}$$

We draw the triangle UVW in red and its standardly rotated image $U'V'W'$ in blue.



Exercise 134

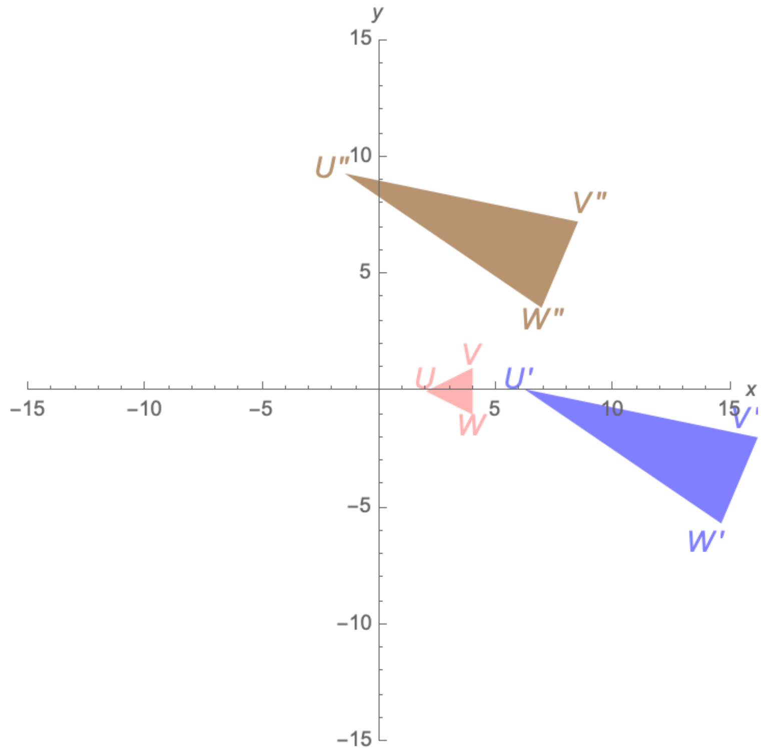
Erratum: *the rotation angle is intended to be* $-23°$ *instead of the positive value.*

Given the translation $\mathrm{T}_{\binom{-3}{4}}$, the standard scaling $\mathrm{S}_{\binom{5}{2}}$ and the standard rotation $\mathrm{R}_O(-23°)$ deliver these composite transformation matrices:

1) the action $\mathrm{A} = \mathrm{T}_{\binom{-3}{4}} \cdot \mathrm{R}_O(-23°) \cdot \mathrm{S}_{\binom{5}{2}} = \begin{pmatrix} 4.60 & 0.78 & -3.00 \\ -1.95 & 1.84 & 4.00 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}$

2) the action $\mathrm{B} = \mathrm{R}_O(-23°) \cdot \mathrm{S}_{\binom{5}{2}} \cdot \mathrm{T}_{\binom{-3}{4}} = \begin{pmatrix} 4.60 & 0.78 & -10.68 \\ -1.95 & 1.84 & 13.23 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}$

Exercise 135 We draw the triangle UVW in red, its A-image $U'V'W'$ in blue and its B-image $U''V''W''$ in brown.



Exercise 136 Retrieving the ingredients of

1) the TRS-based action transformation A,

   - ▷ read the $T_{\begin{pmatrix} -3.00 \\ 4.00 \end{pmatrix}}$ displacement vector from the last matrix column,
   - ▷ calculate column wise the $S_{\begin{pmatrix} s_x \\ s_y \end{pmatrix}}$ scale factors respectively as

$$\begin{aligned} s_x &= \|\vec{v}_1\| = \sqrt{(4.60)^2 + (-1.95)^2} \approx 5.00 \\ s_y &= \|\vec{v}_2\| = \sqrt{(0.78)^2 + (1.84)^2} \approx 2.00 \end{aligned}$$

   - ▷ and finally retrieve the $R_O(\theta)$ rotation angle via the inverse tangent-with-quadrant function atan2 which takes two arguments, given $s_x > 0$

$$\begin{aligned} \theta &= \operatorname{atan2}\left(v_{1_y}, v_{1_x}\right) \\ &= \operatorname{atan2}(-1.95, 4.60) \approx -22.97^\circ \end{aligned}$$

2) the loosely composite action transformation B: fail for translation ingredient. Also, in general for various non-TRS composites, beware of a fail.

Exercise 137 For the ingredients of the given TRS-based action transformation A,

- ▷ read the $T_{\begin{pmatrix} 5.00 \\ 4.00 \end{pmatrix}}$ displacement vector from the last matrix column,
- ▷ calculate column wise the $S_{\begin{pmatrix} s_x \\ s_y \end{pmatrix}}$ scale factors respectively as

$$\begin{aligned} s_x &= \|\vec{v}_1\| = \sqrt{(1.73)^2 + (1.00)^2} \approx 2.00 \\ s_y &= \|\vec{v}_2\| = \sqrt{(-1.50)^2 + (2.60)^2} \approx 3.00 \end{aligned}$$

- ▷ and finally retrieve the $R_O(\theta)$ rotation angle via the inverse tangent-with-quadrant function atan2 which takes two arguments, given $s_x > 0$

$$\begin{aligned} \theta &= \operatorname{atan2}\left(v_{1_y}, v_{1_x}\right) \\ &= \operatorname{atan2}(1.00, 1.73) \approx +30.00^\circ \end{aligned}$$

Exercise 138

To pivot the square $ABCD$ around its centroid $Z = \frac{1}{4}(A+B+C+D) = (1.5\,, 1.5)$ we ‘sandwich’ the standard rotator $\begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) & 0 \\ \sin(45^\circ) & \cos(45^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ by the appropriate translators to and fro the standard position in the origin O.

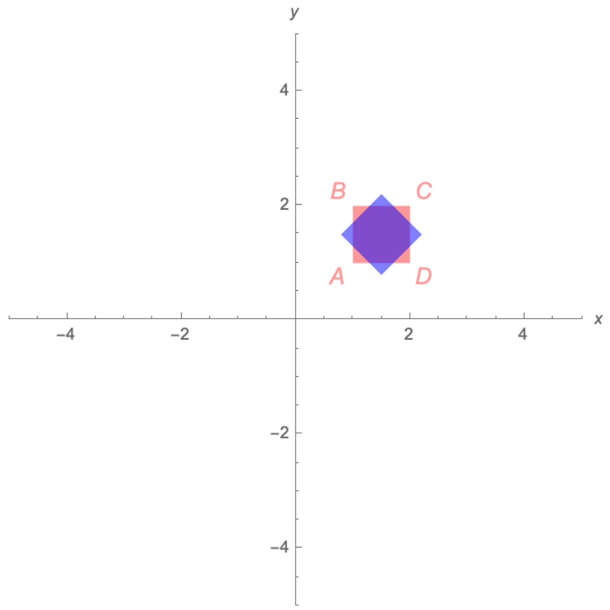
Applying the pivot-operator

$$P_Z(45^\circ) = \begin{pmatrix} 1 & 0 & 1.5 \\ 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) & 0 \\ \sin(45^\circ) & \cos(45^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1.5 \\ 0 & 1 & -1.5 \\ 0 & 0 & 1 \end{pmatrix}$$

onto the vertices of the square $ABCD$ produces the pivoted image square $A'B'C'D'$

$$P_Z(45^\circ) \cdot \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \approx \begin{pmatrix} 1.50 & 0.79 & 1.50 & 2.21 \\ 0.79 & 1.50 & 2.21 & 1.50 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We draw the original square $ABCD$ in red and its pivoted image $A'B'C'D'$ in blue.



Exercise 139

To orbit the square $ABCD$ around the origin O we simply apply the standard rotator.

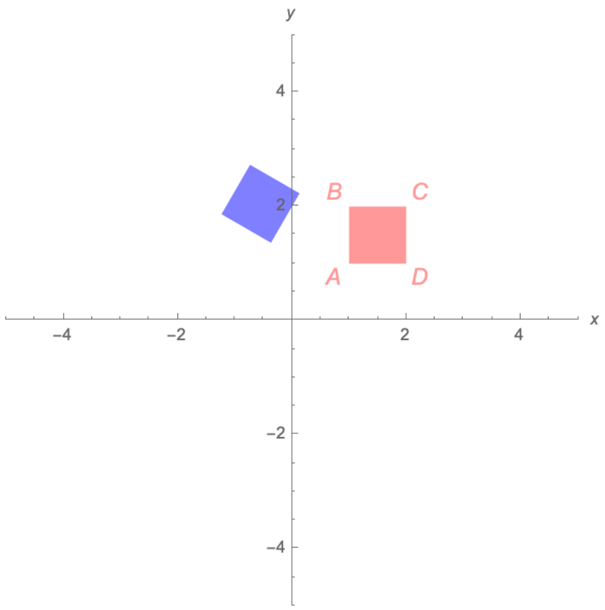
Applying this **standard** orbit-operator

$$O(60^\circ) = \begin{pmatrix} \cos(60^\circ) & -\sin(60^\circ) & 0 \\ \sin(60^\circ) & \cos(60^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

onto the vertices of the square $ABCD$ produces the orbited image square $A''B''C''D''$

$$O(60^\circ) \cdot \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \approx \begin{pmatrix} -0.37 & -1.23 & -0.73 & 0.13 \\ 1.37 & 1.87 & 2.73 & 2.23 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We draw the original square $ABCD$ in red and its orbited image $A''B''C''D''$ in blue.



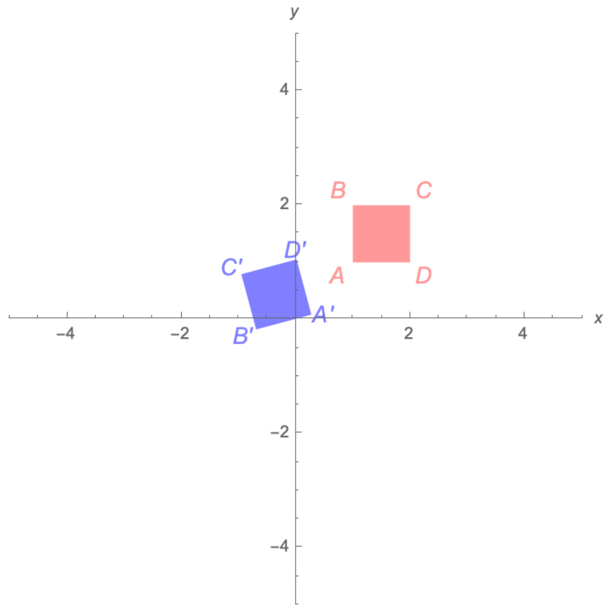
Exercise 140 Applying the composite operator

$$\begin{aligned} G &= P_Z(45^\circ)\cdot O(60^\circ) \\ &= \begin{pmatrix} 1 & 0 & 1.5 \\ 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) & 0 \\ \sin(45^\circ) & \cos(45^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1.5 \\ 0 & 1 & -1.5 \\ 0 & 0 & 1 \end{pmatrix} \cdot O(60^\circ) \end{aligned}$$

onto the vertices of the square $ABCD$ produces the G-image square

$$G\cdot \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \approx \begin{pmatrix} 0.28 & -0.69 & -0.95 & 0.02 \\ 0.09 & -0.17 & 0.79 & 1.05 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We draw the original square $ABCD$ in red and its G-image $A'B'C'D'$ in blue.



Exercise 141

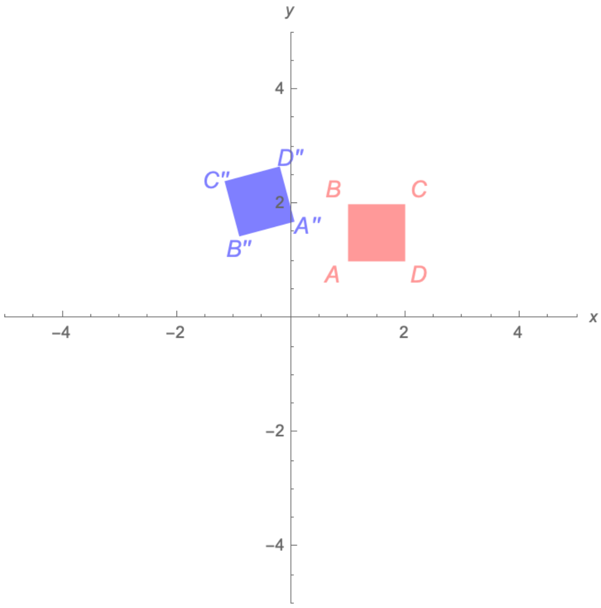
Applying the composite operator

$$\begin{aligned} H &= O(60^\circ)\cdot P_Z(45^\circ) \\ &= O(60^\circ)\cdot\begin{pmatrix} 1 & 0 & 1.5 \\ 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) & 0 \\ \sin(45^\circ) & \cos(45^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & -1.5 \\ 0 & 1 & -1.5 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

onto the vertices of the square $ABCD$ produces the H-image square

$$H\cdot\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\approx\begin{pmatrix} 0.06 & -0.90 & -1.16 & -0.20 \\ 1.70 & 1.44 & 2.40 & 2.66 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We draw the original square $ABCD$ in red and its H-image $A''B''C''D''$ in blue.



Comparing this H-outcome to the G-outcome of the previous exercise 140, reveals how transformations are not commutative. This is of course due to the fact that the matrix product is not commutative.

Exercise 142

1) Computing the look-at action matrix $A_{\overrightarrow{BT}}$ we TRS-wise assemble it as

$$\begin{aligned} A_{\overrightarrow{BT}} &= T_{\overrightarrow{OB}} \cdot R_{\widehat{BT}} \cdot S_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = T_{\overrightarrow{OB}} \cdot R_{\widehat{BT}} \cdot I_3 = T_{\overrightarrow{OB}} \cdot R_{\widehat{BT}} \\ &= T_{\begin{pmatrix} 1 \\ 7 \end{pmatrix}} \cdot R_{\widehat{BT}} \end{aligned}$$

which will point the isosceles triangle KLM situated in B towards a target positioned in T. We therefore need to determine the shape's desired direction vector as

$$\begin{aligned} \hat{v}_1 &= \frac{\overrightarrow{BT}}{\|\overrightarrow{BT}\|} = \frac{1}{\|\overrightarrow{BT}\|}(\vec{t}-\vec{b}) \\ &= \frac{1}{\sqrt{(-10)^2+(-1)^2}}\left(\begin{pmatrix} -9 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 7 \end{pmatrix}\right) = \frac{1}{\sqrt{101}}\begin{pmatrix} -10 \\ -1 \end{pmatrix} \end{aligned}$$

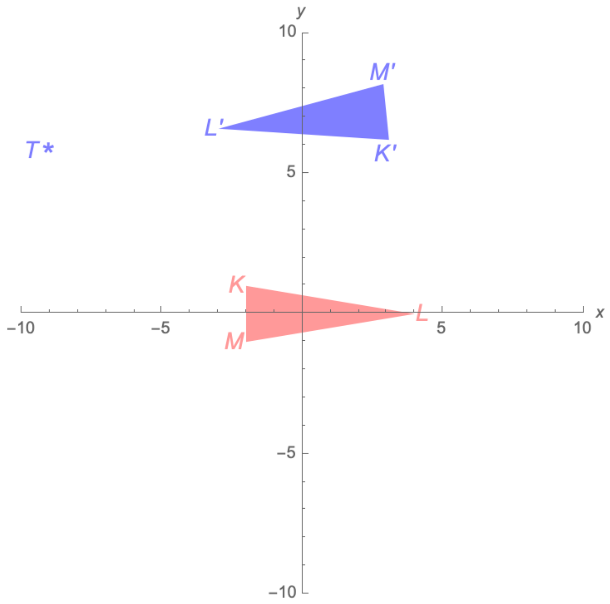
Hence we compose all the above into the look-at operator

$$A_{\overrightarrow{BT}} \approx \begin{pmatrix} -0.995 & 0.099 & 1.000 \\ -0.099 & -0.995 & 7.000 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{given } \hat{v}_1 \approx \begin{pmatrix} -0.995 \\ -0.099 \end{pmatrix}$$

Applying this look-at action onto the vertices of the triangle KLM produces its image pointer $K'L'M'$

$$A_{\overrightarrow{BT}} \cdot \begin{pmatrix} -2 & 4 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \approx \begin{pmatrix} 3.09 & -2.98 & 2.89 \\ 6.20 & 6.60 & 8.19 \\ 1 & 1 & 1 \end{pmatrix}$$

2) We draw the original triangle KLM in red and its image pointer $K'L'M'$ (from centre B looking at target T) in blue.



Exercise 143 To retrieve the ingredients of the look-at action matrix $A_{\overrightarrow{BT}}$ of the previous exercise 143, we

- ▷ read the $T_{\begin{pmatrix} 1 \\ 7 \end{pmatrix}}$ displacement vector from the last matrix column,
- ▷ calculate column wise the $S_{\begin{pmatrix} s_x \\ s_y \end{pmatrix}}$ scale factors respectively as

$$\begin{aligned} s_x &= \|\vec{v}_1\| = \sqrt{(-0.995)^2 + (-0.099)^2} \approx 1.00 \\ s_y &= \|\vec{v}_2\| = \sqrt{(0.099)^2 + (-0.995)^2} \approx 1.00 \end{aligned}$$

▷ and finally retrieve the $R_O(\theta)$ rotation angle via the inverse tangent-with-quadrant function atan2 which takes two arguments, given $s_x > 0$

$$\begin{aligned}\theta &= \operatorname{atan2}\left(v_{1_y}, v_{1_x}\right)\\ &= \operatorname{atan2}(-0.099, -0.995) \approx -174.289^\circ\end{aligned}$$

Exercise 144 Given is the look-at action matrix computed on page 324 as

$$A_{\overrightarrow{BT}} \approx \begin{pmatrix} 0.93 & -0.37 & 5.00 \\ 0.37 & 0.93 & 4.00 \\ 0 & 0 & 1 \end{pmatrix}.$$

To retrieve the ingredients of this look-at action matrix $A_{\overrightarrow{BT}}$, we

▷ read the $T_{\begin{pmatrix} 5 \\ 4 \end{pmatrix}}$ displacement vector from the last matrix column,

▷ calculate column wise the $S_{\begin{pmatrix} s_x \\ s_y \end{pmatrix}}$ scale factors respectively as

$$\begin{aligned}s_x &= \|\vec{v}_1\| = \sqrt{0.93^2 + 0.37^2} \approx 1.00\\ s_y &= \|\vec{v}_2\| = \sqrt{(-0.37)^2 + 0.93^2} \approx 1.00\end{aligned}$$

▷ and finally retrieve the $R_O(\theta)$ rotation angle via the inverse tangent-with-quadrant function atan2 which takes two arguments, given $s_x > 0$

$$\begin{aligned}\theta &= \operatorname{atan2}\left(v_{1_y}, v_{1_x}\right)\\ &= \operatorname{atan2}(0.37, 0.93) \approx +21.80^\circ\end{aligned}$$

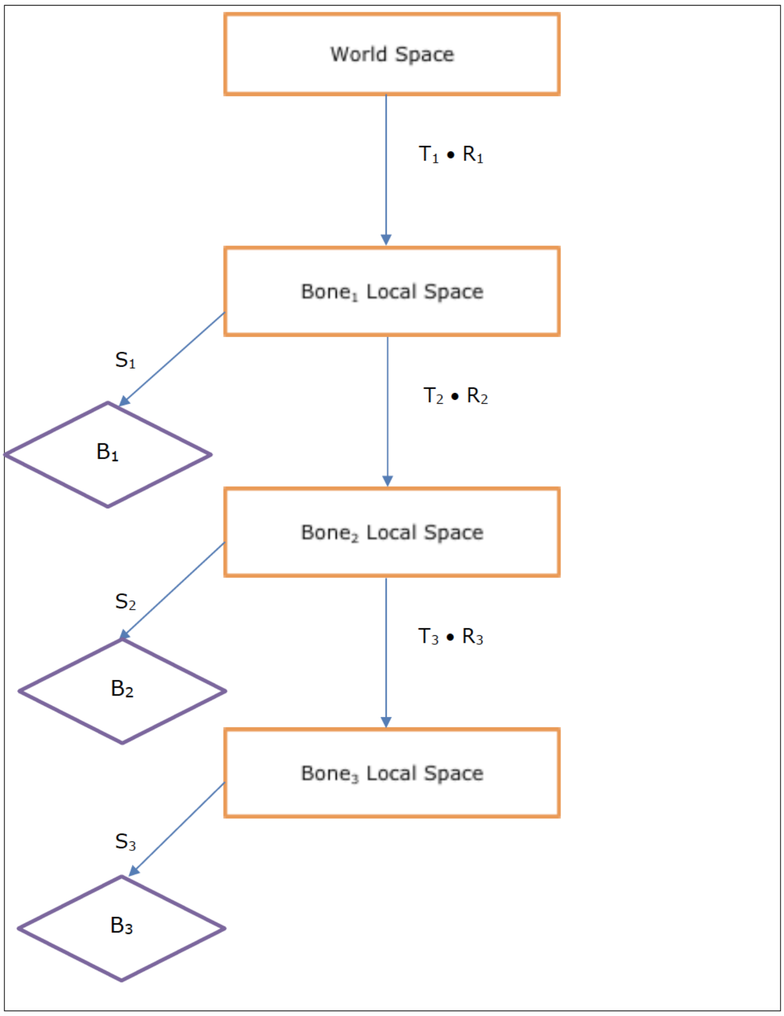
15. Scene Graphs

Exercise 145

We model the bones themselves by scaling the blueprint diamond B_0 according to their required sizes.

$$B_0 = \begin{pmatrix} 0 & 0.5 & 1 & 0.5 \\ 0 & 0.5 & 0 & -0.5 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

1) Firstly, the parent-to-child object tree for this robot arm looks like



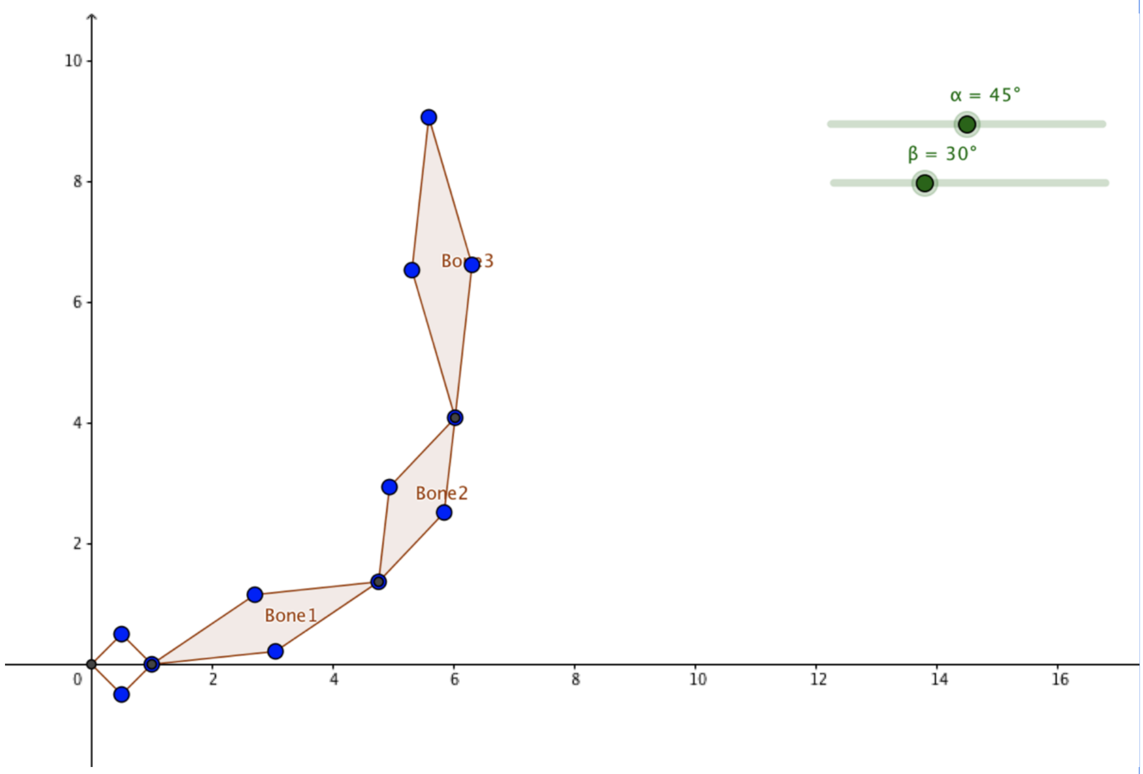
2) Secondly, the described embedding tranformations that link each successive local space of the scene graph read like

▷ $E_1 = \begin{pmatrix} \cos 20^\circ & -\sin 20^\circ & 1 \\ \sin 20^\circ & \cos 20^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix}$ hosting $B_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot B_0$,

▷ $E_2 = \begin{pmatrix} \cos \alpha & -\sin \alpha & 4 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$ hosting limb $B_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot B_0$,

▷ $E_3 = \begin{pmatrix} \cos \beta & -\sin \beta & 3 \\ \sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ hosting limb $B_3 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot B_0$.

3) Finally, implemented in **GeoGebra** the arm looks like this

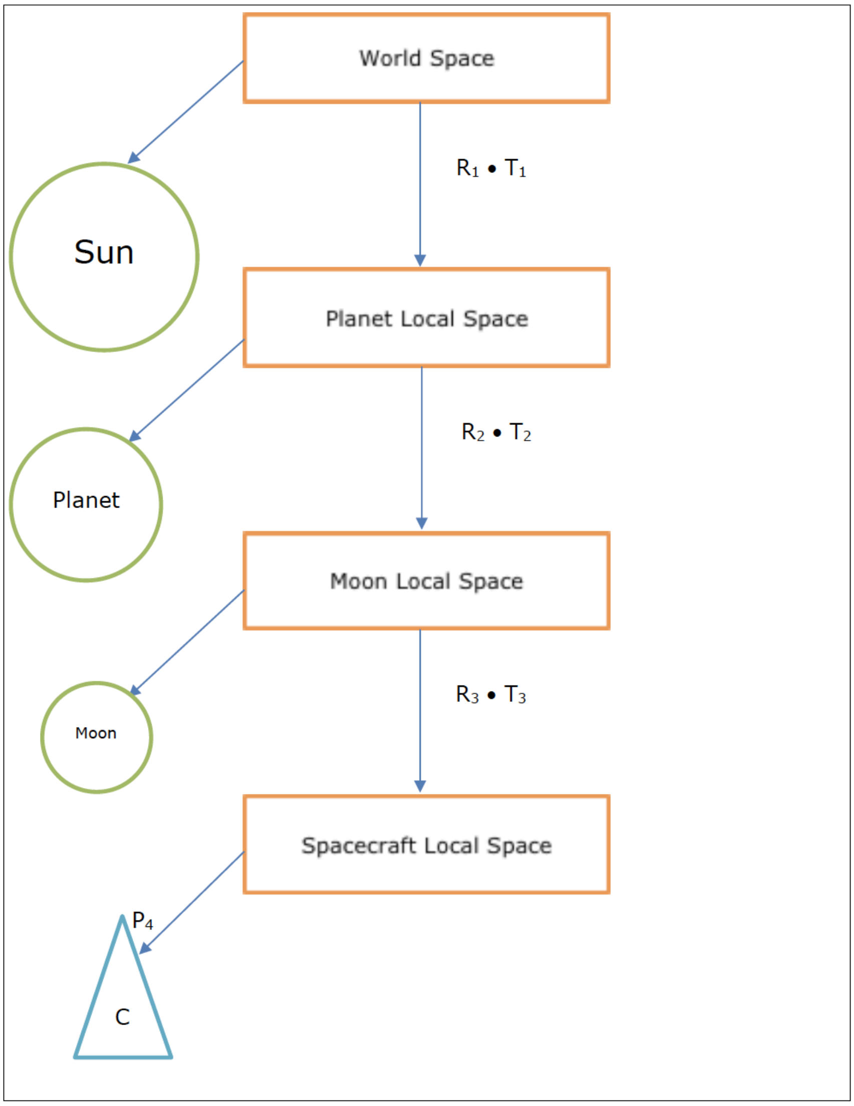


Exercise 146

We use the unit circle $C_0 = C(O;1)$ and model by the isosceles triangle the

$$\text{Craft} \quad = \quad \begin{pmatrix} 1.5 & -1.5 & -1.5 \\ 0.0 & 1.0 & -1.0 \\ 1 & 1 & 1 \end{pmatrix}$$

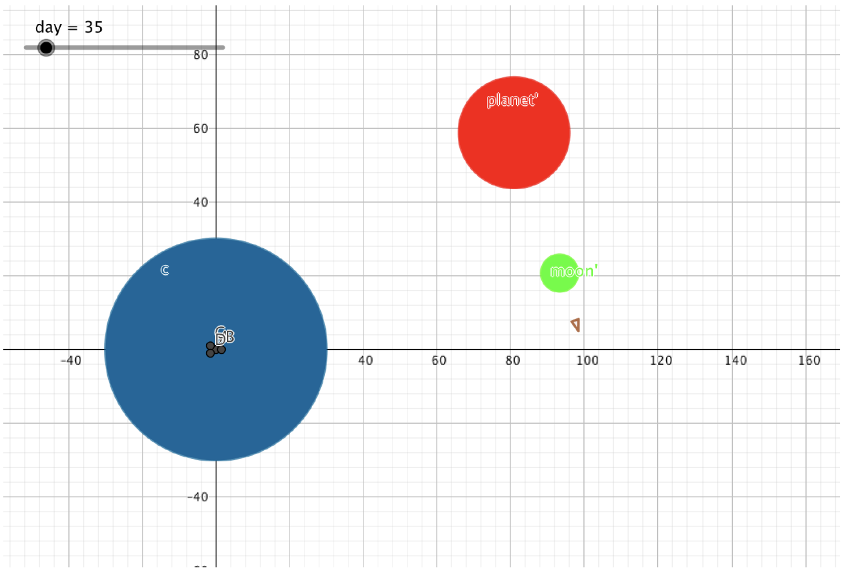
1) Firstly, the parent-to-child object tree for this solar system looks like



2) Secondly, the described embedding tranformations read like

- $E_1 = \begin{pmatrix} \cos\left(\frac{2\pi}{350}day\right) & -\sin\left(\frac{2\pi}{350}day\right) & 0 \\ \sin\left(\frac{2\pi}{350}day\right) & \cos\left(\frac{2\pi}{350}day\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 100 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
  hosting the Planet $= \begin{pmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot C_0$ given the unit circle C_0,
- $E_2 = \begin{pmatrix} \cos\left(\frac{2\pi}{50}day\right) & -\sin\left(\frac{2\pi}{50}day\right) & 0 \\ \sin\left(\frac{2\pi}{50}day\right) & \cos\left(\frac{2\pi}{50}day\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 40 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
  hosting its Moon $= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot C_0$ given the unit circle C_0,
- $E_3 = \begin{pmatrix} \cos\left(\frac{2\pi}{5}day\right) & -\sin\left(\frac{2\pi}{5}day\right) & 0 \\ \sin\left(\frac{2\pi}{5}day\right) & \cos\left(\frac{2\pi}{5}day\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 15 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
  hosting the satellite Craft $= \begin{pmatrix} 1.5 & -1.5 & -1.5 \\ 0.0 & 1.0 & -1.0 \\ 1 & 1 & 1 \end{pmatrix}$.

3) Finally, implemented in **GeoGebra** this solar system looks like this

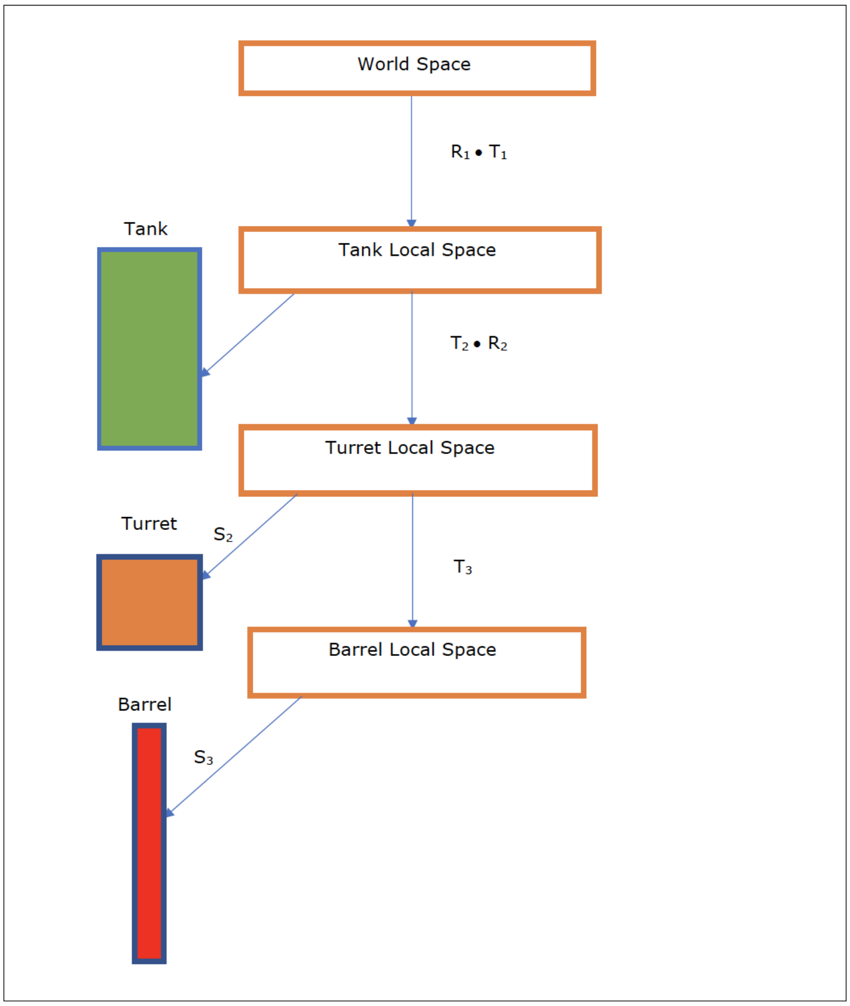


Exercise 147

The rectangular blueprint T_0 becomes the *tank*. The blueprint T_0 scales non-uniformly by scale factor $s_y = 0.5$ to its square *turret*. Also T_0 scales non-uniformly by scale factor $s_x = 0.25$ to a stretched *barrel*.

$$T_0 = \begin{pmatrix} -1 & 1 & 1 & -1 \\ 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

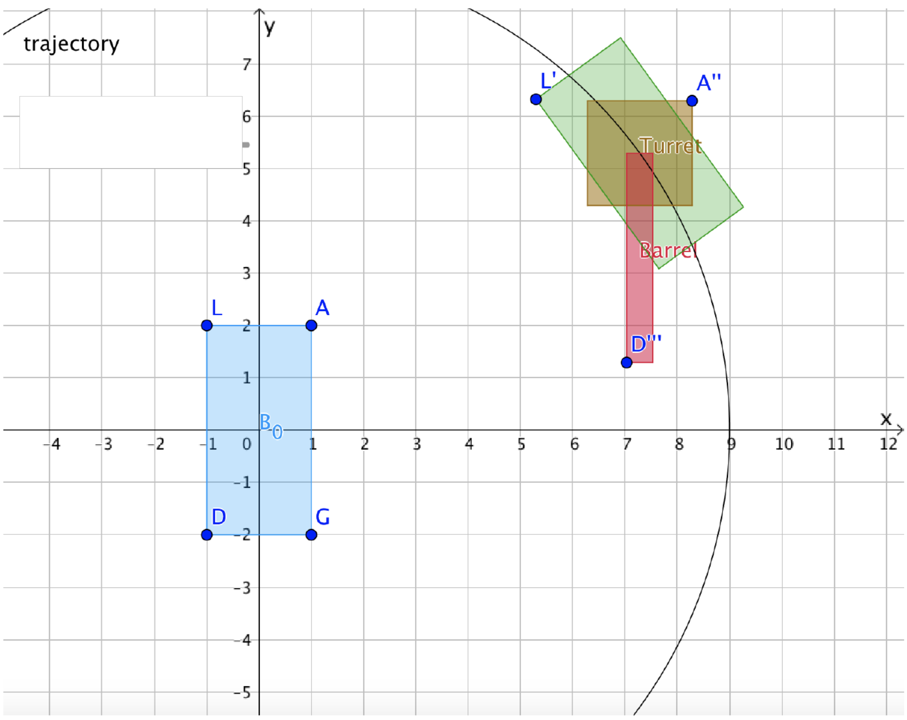
1) The parent-to-child object tree for this layered object *tank-turret-barrel* is



2) Secondly, the described embedding tranformations read like

- ▷ $E_1 = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ hosting the Tank T_0 with its orbit center positioned at a freely chosen distance 9 (in point O),
- ▷ $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ hosting, set $\beta = -\alpha$ to keep its orientation to the South, the Turret $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot T_0$,
- ▷ $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ hosting its fixed Barrel $= \begin{pmatrix} 1 & 0.25 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot T_0$.

3) Finally, implemented in **GeoGebra** this layered *tank-turret-barrel* system looks like this



16. View Transformation

Exercise 148 Given the world space is 100 pixels wide and 50 high and the camera window is 15 wide and 15 high, and is rotated over a 15° angle around its bottom left vertex C in position $(87,\ 10)$.

Performing the required boundary check, takes

1) constructing the camera transformation,

$$\begin{aligned} F_{\binom{87}{10}}(15^\circ) &= T_{\binom{87}{10}} \cdot R_O(15^\circ) \cdot S_{\binom{1}{1}} \\ &= \begin{pmatrix} \cos 15^\circ & -\sin 15^\circ & 87 \\ \sin 15^\circ & \cos 15^\circ & 10 \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 0.97 & -0.26 & 87 \\ 0.26 & 0.97 & 10 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

2) determining the vertices of the camera window,

$$Cam = \begin{pmatrix} 0 & 15 & 15 & 0 \\ 0 & 0 & 15 & 15 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

3) transforming the camera window vertices by the camera transformation,

$$\begin{aligned} Cam' &= F_{\binom{87}{10}}(15^\circ) \cdot Cam \\ &\approx \begin{pmatrix} 87.00 & 101.50 & 97.61 & 83.12 \\ 10.00 & 13.88 & 28.37 & 24.49 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{aligned}$$

4) verifying wether the camera window image Cam' stays within the world boundaries, we discover the second image vertex $(101.50,\ 13.88)$ to be outside of the given world 100×50-rectangle.

Exercise 149 A rectangular camera window has a width of 10 by a height of 8 units. If the left bottom vertex C of the rectangle is located at the point $(3,\ 4)$ and it is rotated over an angle of 30° around C then we determine the

1) camera transformation to put the camera window at this position as,

$$\begin{aligned} F_{\binom{3}{4}}(30^\circ) &= T_{\binom{3}{4}} \cdot R_O(30^\circ) \quad \text{without any scaling yet} \\ &= \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ & 3 \\ \sin 30^\circ & \cos 30^\circ & 4 \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 0.87 & -0.50 & 3 \\ 0.50 & 0.87 & 4 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

2) view transformation to return the camera capture horizontally filling our fixed game window measuring 800 by 600 pixels as

$$
\begin{aligned}
V_{\binom{3}{4}}(30^\circ) &= F_{\binom{3}{4}}(30^\circ)^{-1} \\
&= \left(T_{\binom{3}{4}} \cdot R_O(30^\circ) \cdot S_{\binom{s_x}{s_y}}\right)^{-1} \\
&= S^{-1}_{\binom{s_x}{s_y}} \cdot R_O^{-1}(30^\circ) \cdot T^{-1}_{\binom{3}{4}} \\
&= S_{\binom{1/s_x}{1/s_y}} \cdot R_O(-30^\circ) \cdot T_{\binom{-3}{-4}} \\
&= \begin{pmatrix} \frac{800}{10} & 0 & 0 \\ 0 & \frac{600}{8} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(-30^\circ) & -\sin(-30^\circ) & 0 \\ \sin(-30^\circ) & \cos(-30^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 80 & 0 & 0 \\ 0 & 75 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos 30^\circ & \sin 30^\circ & 0 \\ -\sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \\
&\approx \begin{pmatrix} 80 & 0 & 0 \\ 0 & 75 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.87 & 0.50 & -4.6 \\ -0.50 & 0.87 & 1.96 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 69.28 & 40.00 & -367.85 \\ -37.50 & 64.95 & -147.31 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Exercise 150

▷ Verifying that the matrix product of matrices (16.1) and (16.6) returns matrix I_3

$$
\begin{aligned}
& F_{\binom{400}{100}}(21.8^\circ) \cdot V_{\binom{400}{100}}(21.8^\circ) \\
\approx\ & \begin{pmatrix} 0.93 & -0.37 & 400.00 \\ 0.37 & 0.93 & 100.00 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.93 & 0.37 & -409.00 \\ -0.37 & 0.93 & 55.70 \\ 0 & 0 & 1 \end{pmatrix} \\
\approx\ & \begin{pmatrix} 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

▷ Proving that the matrix product matrices (16.2) and (16.4) yields matrix I_3

$$\begin{pmatrix} s_x\cos\theta & -s_y\sin\theta & c_1 \\ s_x\sin\theta & s_y\cos\theta & c_2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\cos\theta}{s_x} & \frac{\sin\theta}{s_x} & \frac{-\hat{v}_1\cdot\vec{c}}{s_x} \\ -\frac{\sin\theta}{s_y} & \frac{\cos\theta}{s_y} & \frac{-\hat{v}_2\cdot\vec{c}}{s_y} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (\cos\theta)^2+(\sin\theta)^2 & 0 & -((\cos\theta)^2+(\sin\theta)^2)c_1+c_1 \\ 0 & (\cos\theta)^2+(\sin\theta)^2 & -((\sin\theta)^2+(\cos\theta)^2)c_2+c_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

▷ Proving the matrix product of matrices (16.3) and (16.5) produces matrix I_3

$$F_{\vec{c}}(\theta)\cdot V_{\vec{c}}(\theta) = \begin{pmatrix} v_{1_x} & v_{2_x} & c_1 \\ v_{1_y} & v_{2_y} & c_2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\hat{v}_{1_x}}{\|\vec{v}_1\|} & \frac{\hat{v}_{1_y}}{\|\vec{v}_1\|} & -\frac{\hat{v}_1\cdot\vec{c}}{\|\vec{v}_1\|} \\ \frac{\hat{v}_{2_x}}{\|\vec{v}_2\|} & \frac{\hat{v}_{2_y}}{\|\vec{v}_2\|} & -\frac{\hat{v}_2\cdot\vec{c}}{\|\vec{v}_2\|} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (\hat{v}_{1_x})^2+(\hat{v}_{2_x})^2 & 0 & 0 \\ 0 & (\hat{v}_{1_y})^2+(\hat{v}_{2_y})^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (\cos\theta)^2+(-\sin\theta)^2 & 0 & 0 \\ 0 & (\sin\theta)^2+(\cos\theta)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

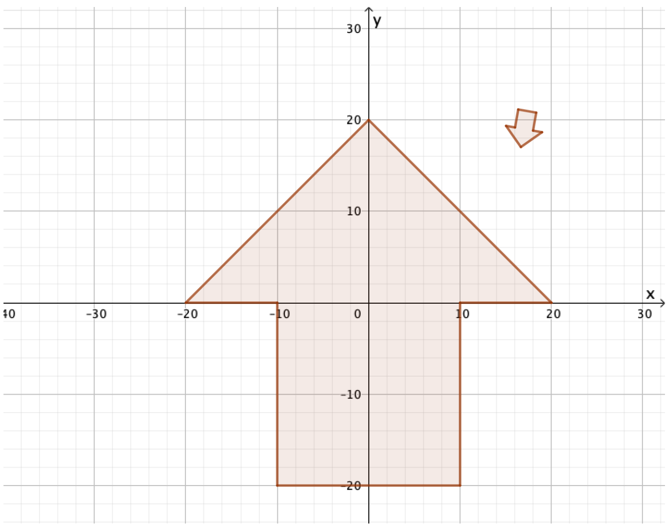
Exercise 151 Construct an arrow spanning edges between vertices, in their order $\{(0,\ 20),(20,\ 0),(10,\ 0),(10,\ -20),(-10,\ -20),$ **(-10, 0),**$(-20,\ 0),(0,\ 20)\}$.

1) Positioning this arrow in the world space implies we need to construct its embedding transformation E. **First scale it to $\frac{1}{10}$ of its original size, then rotate it by an angle of $+170^\circ$ around its pivot and finally position this instanced arrow with its pivot put in point $(17,\ 19)$.** Calculating and visualising the according arrow image positioned in the world space in GeoGebra by setting

$$A_0 = \begin{pmatrix} 0 & 20 & 10 & 10 & -10 & -10 & -20 & 0 \\ 20 & 0 & 0 & -20 & -20 & 0 & 0 & 20 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

upon which we TRS-place the arrow A_0 in world space as pictured underneath.

$$\begin{aligned} E &= \begin{pmatrix} 1 & 0 & 17 \\ 0 & 1 & 19 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos 170^\circ & -\sin 170^\circ & 0 \\ \sin 170^\circ & \cos 170^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\approx \begin{pmatrix} -0.099 & -0.017 & 17 \\ 0.017 & -0.099 & 19 \\ 0 & 0 & 1 \end{pmatrix} \\ A_0' &= E \cdot A_0 \\ &\approx \begin{pmatrix} 16.65 & 15.03 & 16.02 & 16.36 & 18.33 & 17.98 & 18.97 & 16.65 \\ 17.03 & 19.35 & 19.17 & 21.14 & 20.80 & 18.83 & 18.65 & 17.03 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{aligned}$$

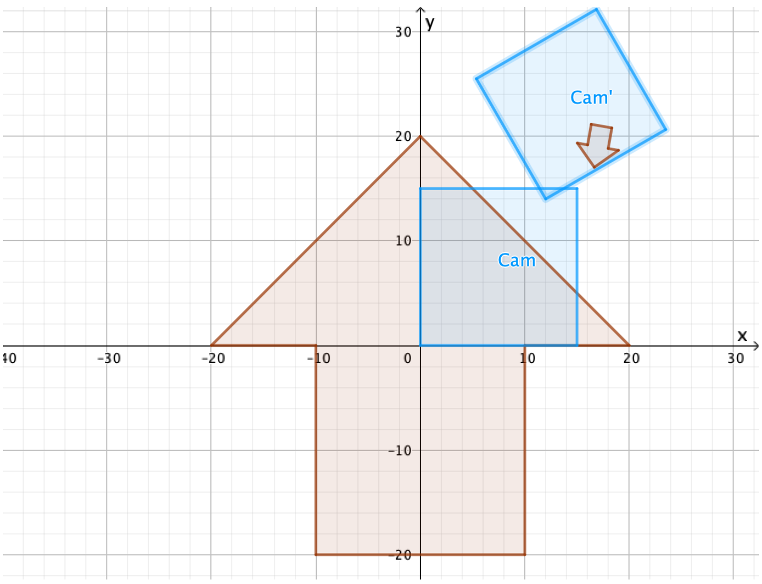


2) Returning the camera captured arrow from world space to the fixed game window, which is requiring a view transformation. Calculating and visualising the according camera capture positioned in the world space in GeoGebra by setting

$$Cam = \begin{pmatrix} 0 & 0 & 15 & 15 & 0 \\ 0 & 15 & 15 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

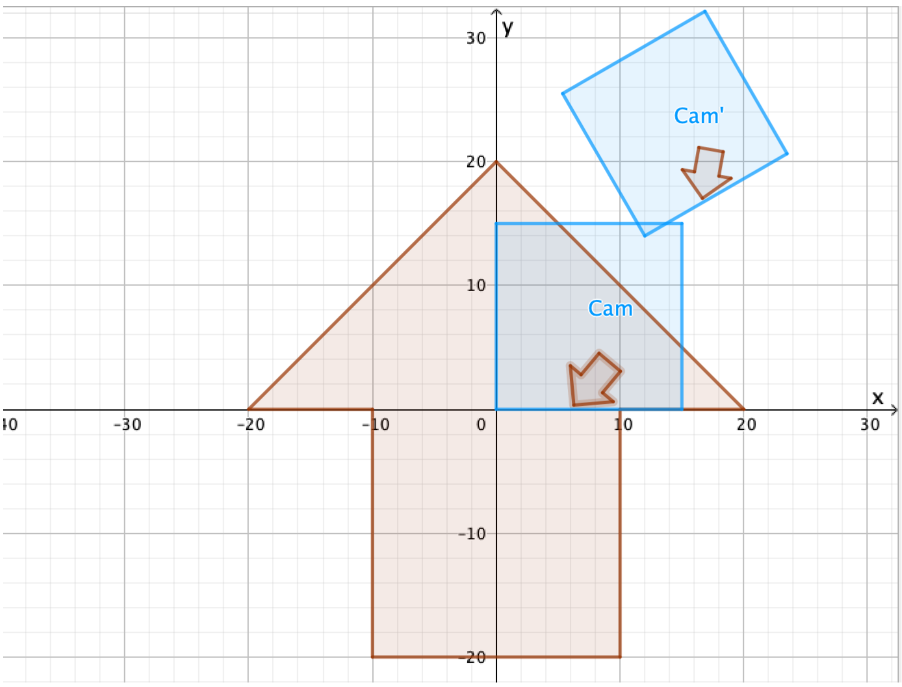
upon which we forwardly TRS-place the camera *Cam* in world space by the camera transformation *F* as pictured underneath.

$$\begin{aligned} F &= \begin{pmatrix} 1 & 0 & 12 \\ 0 & 1 & 14 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{1.13} & 0 & 0 \\ 0 & \frac{1}{1.13} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\approx \begin{pmatrix} 0.77 & -0.44 & 12 \\ 0.44 & 0.77 & 14 \\ 0 & 0 & 1 \end{pmatrix} \\ Cam' &= F \cdot Cam \\ &\approx \begin{pmatrix} 12 & 5.36 & 16.86 & 23.50 & 12 \\ 14 & 25.50 & 32.13 & 20.64 & 14 \\ 1 & 1 & 1 & 1 & .1 \end{pmatrix} \end{aligned}$$

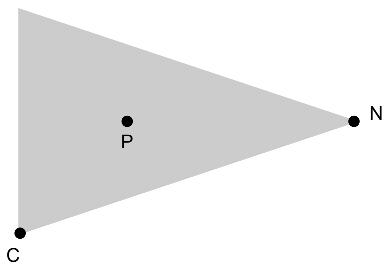


3) Stacking the former view transformation V on top of the latter embedding transformation E, to matrix multiply these into one action matrix. Calculating and visualising the subsequent arrow image A_0'' as brought into the game window in GeoGebra.

$$\begin{aligned}
A_0'' &= V \cdot E \cdot A_0 \\
&\approx \begin{pmatrix} 0.77 & -0.44 & 12 \\ 0.44 & 0.77 & 14 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -0.099 & -0.017 & 17 \\ 0.017 & -0.099 & 19 \\ 0 & 0 & 1 \end{pmatrix} \cdot A_0 \\
&\approx \begin{pmatrix} 0.98 & 0.57 & -19.65 \\ -0.57 & 0.98 & -6.92 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -0.099 & -0.017 & 17 \\ 0.017 & -0.099 & 19 \\ 0 & 0 & 1 \end{pmatrix} \cdot A_0 \\
&\approx \begin{pmatrix} 0.98 & 0.57 & -19.65 \\ -0.57 & 0.98 & -6.92 \\ 0 & 0 & 1 \end{pmatrix} \cdot A_0' \\
&\approx \begin{pmatrix} 6.27 & 5.99 & 6.85 & 8.31 & 10.04 & 8.58 & 9.45 & 6.27 \\ 0.34 & 3.52 & 2.80 & 4.53 & 3.07 & 1.34 & 0.62 & 0.34 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}
\end{aligned}$$

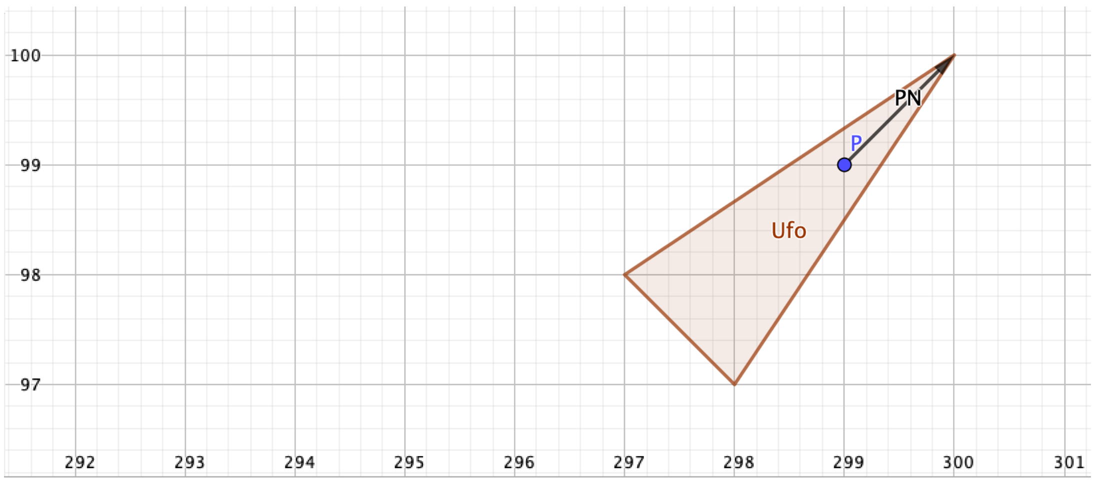


Exercise 152 At a certain time, the pivot point P of such a triangular UFO was located in $(299, 99)$ when its nose point N was in $(300, 100)$. The military camera window measured 15 width by 10 height with the bottom left vertex C of the camera's rectangle situated in $(\mathbf{298}, 97)$ on the UFO's right wing tip as portrayed.



We model this UFO by the isosceles triangle

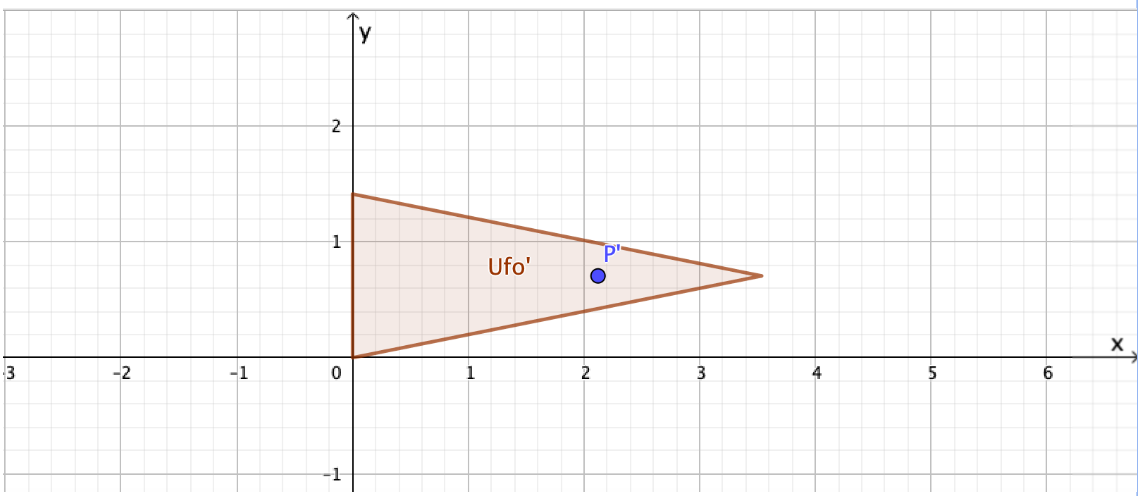
$$U_0 = \begin{pmatrix} 300 & 298 & 297 \\ 100 & 97 & 98 \\ 1 & 1 & 1 \end{pmatrix}$$



1) Given all above, we answer the matrix product to deliver the view transformation which brings this UFO horizontally (meaning $\overrightarrow{PN}$ horizontally) into our game window.

2) We compute the former matrix product using GeoGebra to answer the UFO view transformation matrix.

$$
\begin{aligned}
V &= \begin{pmatrix} \cos(-45^\circ) & -\sin(-45^\circ) & 0 \\ \sin(-45^\circ) & \cos(-45^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -298 \\ 0 & 0 & -97 \\ 0 & 0 & 1 \end{pmatrix} \\
&\approx \begin{pmatrix} 0.71 & 0.71 & -279.31 \\ -0.71 & 0.71 & 142.13 \\ 0 & 0 & 1 \end{pmatrix} \\
U_0' &= V \cdot U_0 \\
&\approx \begin{pmatrix} 2.12 & 0.00 & 0.00 \\ 0.71 & 0.00 & 1.41 \\ 1 & 1 & 1 \end{pmatrix}
\end{aligned}
$$



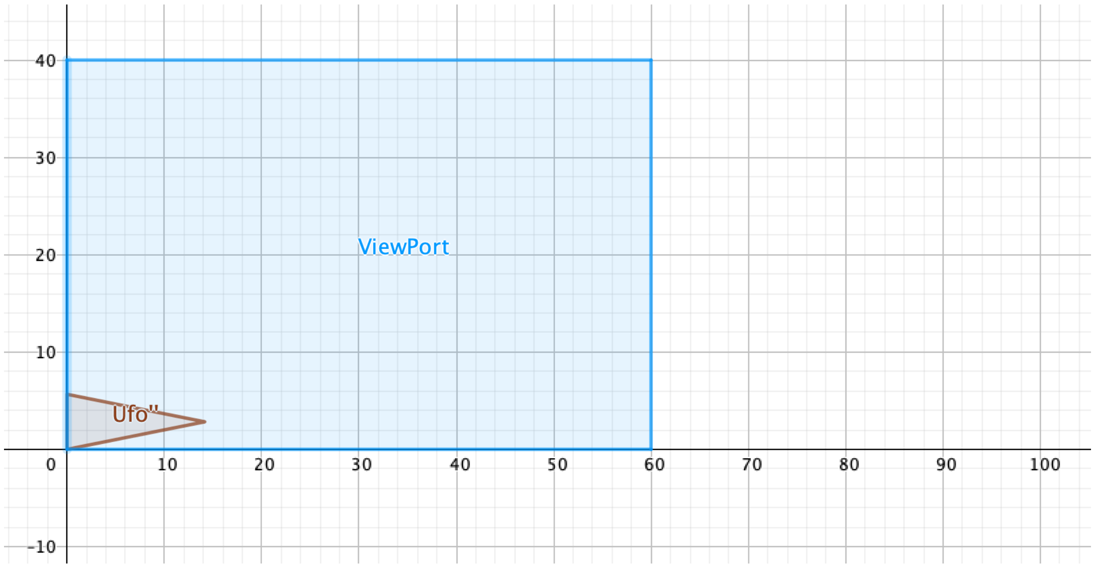
3) We also append a window-filling scaling for a ground based view port of **60** width by **40** height. This scaling is uniform. We recompute the former matrix product now including the latter scaling using GeoGebra to answer the zoomed UFO view transformation matrix.

$$
S = \begin{pmatrix} \frac{60}{15} & 0 & 0 \\ 0 & \frac{40}{10} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$\begin{aligned} U_0'' &= S \cdot U_0' \\ &\approx \begin{pmatrix} 8.49 & 0.00 & 0.00 \\ 2.83 & 0.00 & 5.66 \\ 1 & 1 & 1 \end{pmatrix} \end{aligned}$$

And as for the zoomed UFO view transformation matrix, we conclude

$$\begin{aligned} V_{\text{zoom}} &= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(-45^\circ) & -\sin(-45^\circ) & 0 \\ \sin(-45^\circ) & \cos(-45^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -298 \\ 0 & 0 & -97 \\ 0 & 0 & 1 \end{pmatrix} \\ &\approx \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.71 & 0.71 & -279.31 \\ -0.71 & 0.71 & 142.13 \\ 0 & 0 & 1 \end{pmatrix} \\ &\approx \begin{pmatrix} 2.83 & 2.83 & -1117.23 \\ -2.83 & 2.83 & 568.51 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$



17. Hypercomplex numbers

Exercise 153

$z_1 = 2+2i = \sqrt{8}(\cos 45^\circ + i\sin 45^\circ)$
$z_2 = -1+i = \sqrt{2}(\cos 135^\circ + i\sin 135^\circ)$
$z_3 = -1-\sqrt{3}i = 2(\cos 240^\circ + i\sin 240^\circ)$
$z_4 = 3-3i = \sqrt{18}(\cos 315^\circ + i\sin 315^\circ)$

Exercise 154

1) $-9+3i$
2) $16-24i$
3) $-1+5i$
4) $2-10i$
5) $31-25i$
6) $-1+3i$

Exercise 155

1) $z = \cos 120^\circ + i\sin 120^\circ$
Via the complex number a referring to the point A, we calculate the image vertices (as $a\cdot z, b\cdot z, c\cdot z$ en $d\cdot z$):

$$A' = \left(\frac{-1-\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}\right)$$

$B' = (-\sqrt{3}, 1)$
$C' = (0,2)$

$$D' = \left(\frac{-1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right)$$

2) A complete identical approach applies to $r = 3(\cos 80^\circ + i\sin 80^\circ)$

Exercise 156

$$f(-1) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{-1}}{\sqrt{5}} = \frac{\frac{2}{1+\sqrt{5}} - \frac{2}{1-\sqrt{5}}}{\sqrt{5}} = \frac{\frac{2-2\sqrt{5}-2-2\sqrt{5}}{(1+\sqrt{5})(1-\sqrt{5})}}{\sqrt{5}} = \frac{\frac{-4\sqrt{5}}{1-5}}{\sqrt{5}} = 1$$

Analoguously we calculate $f(-2) = -1, f(-3) = 2, f(-4) - 3, f(-5) = 5, \ldots$

Exercise 157

1) $3-2i-j+4k$
2) 2
3) $2+4i-2k$
4) $4-7i-4j+k$
5) $-2-10i-10j-6k$
6) $\frac{-1}{6}i - \frac{1}{3}j + \frac{1}{6}k$

Exercise 158 $\left[\cos\frac{\theta}{2},\left(0,0,\sin\frac{\theta}{2}\right)\right][0,(x,y,z)]\left[\cos\frac{\theta}{2},\left(0,0,-\sin\frac{\theta}{2}\right)\right]$
$=\left[-z\sin\frac{\theta}{2},\left(x\cos\frac{\theta}{2}-y\sin\frac{\theta}{2},y\cos\frac{\theta}{2}+x\sin\frac{\theta}{2},z\cos\frac{\theta}{2}\right)\right]\left[\cos\frac{\theta}{2},\left(0,0,-\sin\frac{\theta}{2}\right)\right]$

$=[0,(x\cos a-y\sin a,y\cos a+x\sin a,z)]$

Exercise 159

1) $$\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

2) $qrq^*=\left[0,(0,1,0)\right]$

Exercise 160

1) $$\begin{pmatrix}1&0&0&0\\0&\frac{-1}{2}&\frac{\sqrt{3}}{2}&0\\0&\frac{-\sqrt{3}}{2}&\frac{-1}{2}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}2\\3\\1\\1\end{pmatrix}=\begin{pmatrix}2\\\dfrac{-3+\sqrt{3}}{2}\\\dfrac{-1-3\sqrt{3}}{2}\\1\end{pmatrix}$$

2) $qrq^*=\left[0,\left(2,\dfrac{-3+\sqrt{3}}{2},\dfrac{-1-3\sqrt{3}}{2}\right)\right]$

Exercise 161
We calculate $qrq^*=\left[0,\left(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)\right][0,(1,1,0)]\left[0,\left(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)\right]^*=[0,(0,1,1)]$.

The image point equals $(0,1,1)$.

Exercise 162
$Z+Z^*=2a\begin{pmatrix}1&0\\0&1\end{pmatrix}$
$Z-Z^*=-2b\begin{pmatrix}0&-1\\1&0\end{pmatrix}$
$Z\cdot Z^*=\begin{pmatrix}a&-b\\b&a\end{pmatrix}\cdot\begin{pmatrix}a&b\\-b&a\end{pmatrix}=\begin{pmatrix}a^2+b^2&0\\0&a^2+b^2\end{pmatrix}=(a^2+b^2)\begin{pmatrix}1&0\\0&1\end{pmatrix}$

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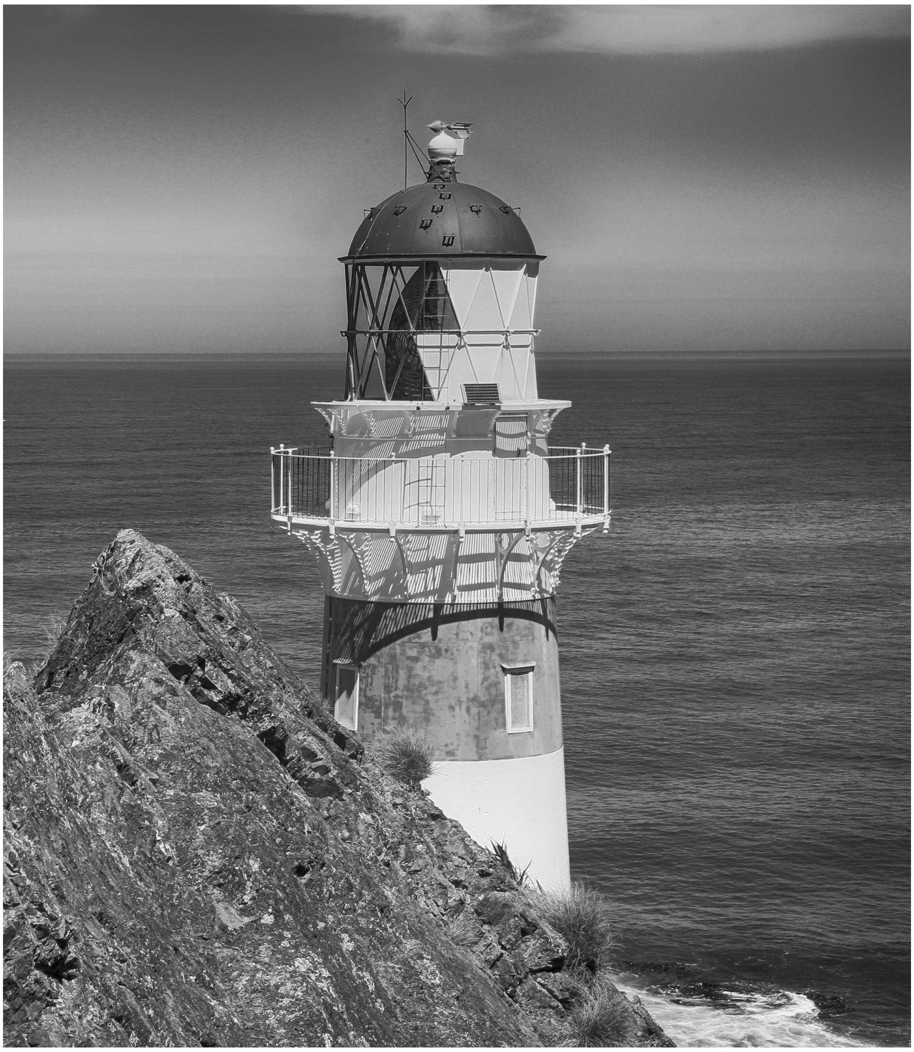
Annex F · Animation Maths (2021) Errata



Lighthouse photograph by Detje Holger at pixabay.com

3. Trigonometry

DELETION OF 'COTERMINAL ANGLES'

F.1 Pairs of angles

We briefly explain the properties of two pairs of angles that are useful in this book.

Oppositely signed angles; their measurements add up to 0°. In other words, if α and β are **opposite** then $\alpha+\beta=0^\circ$ or $\beta=-\alpha$. The corresponding figure shows how the cosines of **opposite** angles remain invariant, while their sines receive opposite signs. This leads to the trigonometric formulas $\cos(-\alpha)=\cos\alpha$ and $\sin(-\alpha)=-\sin\alpha$.

Complementary angles; their measurements add up to 90°. In other words, if α and β are complementary then $\alpha+\beta=90^\circ$ or $\beta=90^\circ-\alpha$. The corresponding figure shows how the sine of α equals the cosine of $90^\circ-\alpha$ and the cosine of α equals the sine of $90^\circ-\alpha$. This leads to the trigonometric formulas $\cos(90^\circ-\alpha)=\sin\alpha$ and $\sin(90^\circ-\alpha)=\cos\alpha$.

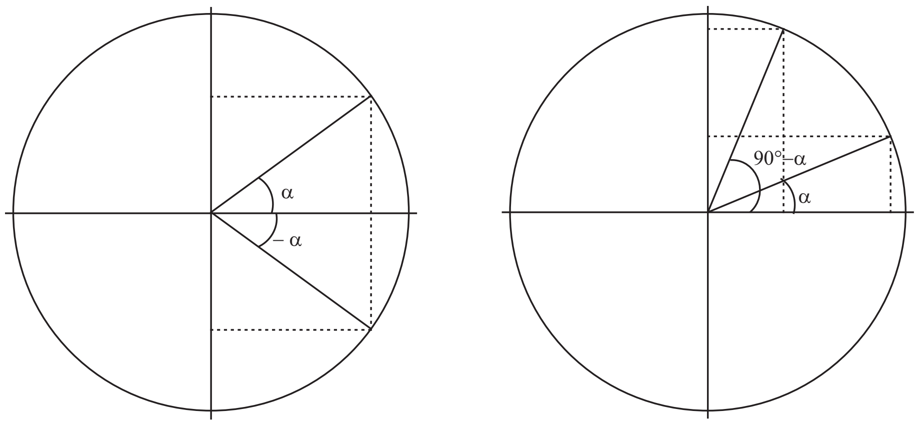


Figure F.1: Opposite and complementary angles

F.2 Sum identities

In this paragraph we state and prove all trigonometric ratios of a sum of two angles. We firstly emphasise the non-linearity of all trigonometric ratios: e.g. for the sine we encounter $\sin(\alpha+\beta) \neq \sin\alpha + \sin\beta$. Indeed, e.g. for angles $\alpha = 60^\circ$ and $\beta = 30^\circ$ the value $\sin 90^\circ = 1$ does not equal the sum $\sin 60^\circ + \sin 30^\circ = \frac{\sqrt{3}+1}{2}$. Given the above inequality, we realise the need for the correct formulas which are stated below.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \tfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta) = \tfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

- Based upon $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, we have less difficulties in proving the five remaining Sum Identities

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos(\alpha-(-\beta)) \\
&= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \text{(\textbf{opposite} angles)}
\end{aligned}$$

$$\begin{aligned}
\sin(\alpha-\beta) &= \cos(90^\circ-(\alpha-\beta)) \qquad \text{(complementary angles)} \\
&= \cos((90^\circ-\alpha)+\beta) \\
&= \cos(90^\circ-\alpha)\cos\beta - \sin(90^\circ-\alpha)\sin\beta \\
&= \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad \text{(complementary angles)}
\end{aligned}$$

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin(\alpha-(-\beta)) \\
&= \sin\alpha\cos(-\beta) - \cos\alpha\sin(-\beta) \\
&= \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \text{(\textbf{opposite} angles)}
\end{aligned}$$

$$\begin{aligned}
\tan(\alpha\pm\beta) &= \frac{\sin(\alpha\pm\beta)}{\cos(\alpha\pm\beta)} \\
&= \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta} \\
&= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} \pm \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} \mp \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\
&= \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta} \qquad \blacksquare
\end{aligned}$$

16. View Transformation

MISSING PICTURES

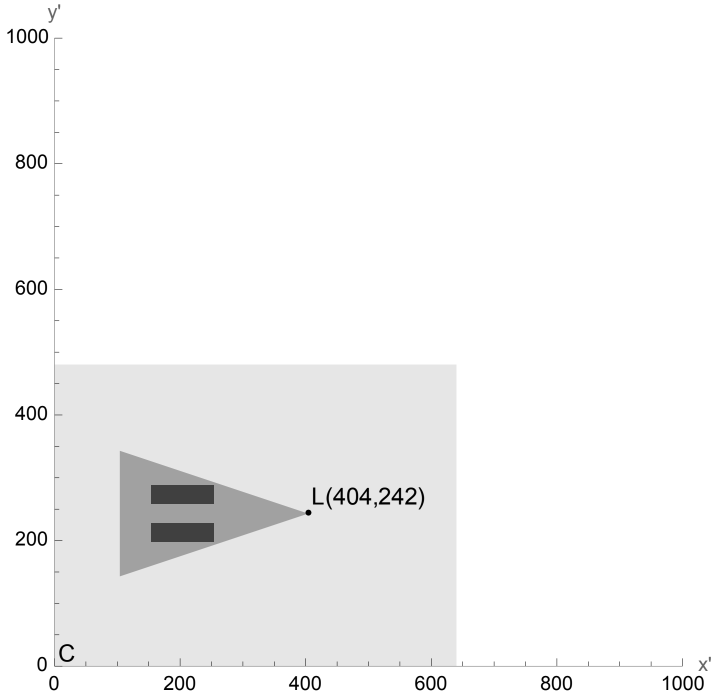


Figure 16.4: The game window

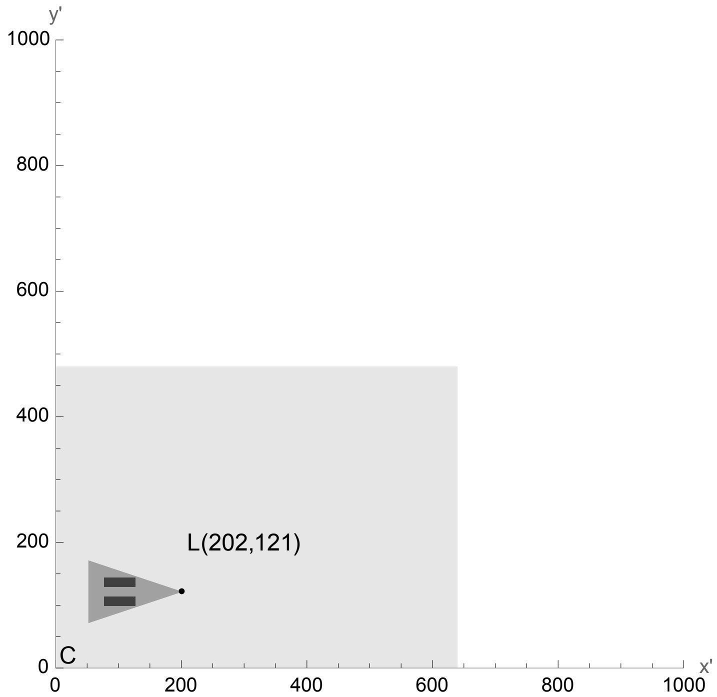


Figure 16.5: Zooming-out view transformation

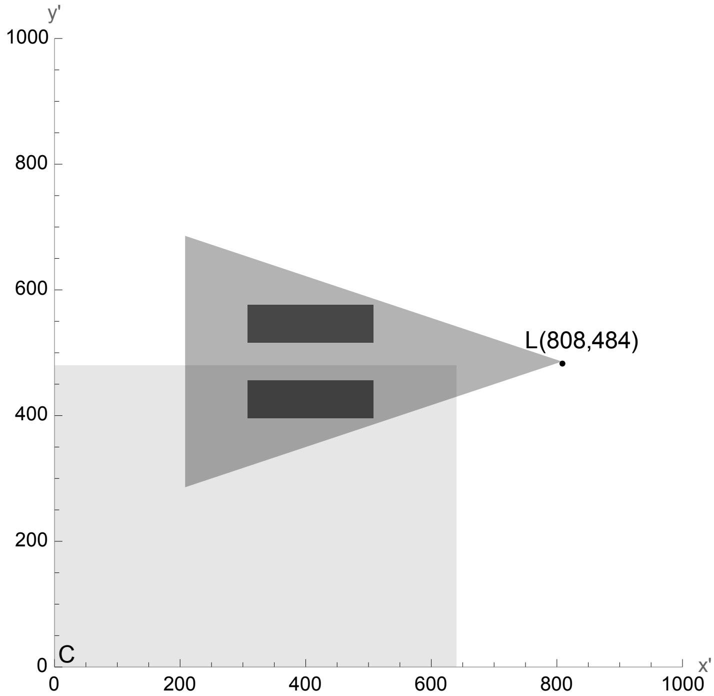


Figure 16.6: Zooming-in view transformation

EXERCISES

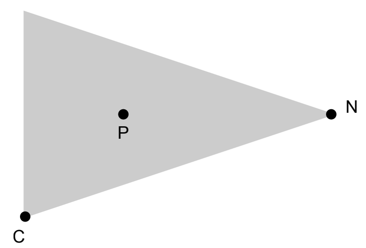
Exercise 151 Construct an arrow spanning edges between vertices, in their order $\{(0,\ 20), (20,\ 0), (10,\ 0), (10,\ -20), (-10,\ -20),$ **(-10, 0),**$(-20,\ 0), (0,\ 20)\}$. We modelled this **arrow** around the origin O which we adopt as its pivot point.

1) Positioning this arrow in the world space implies we need to construct its embedding transformation E. **First scale it to $\frac{1}{10}$ of its original size, then rotate it by an angle of $+170°$ around its pivot and finally position this instanced arrow with its pivot put in point** $(17,\ 19)$**.** Calculate and visualise the according arrow image positioned in the world space in GeoGebra.

2) Return the camera captured arrow from world space to the fixed game window, which is requiring a view transformation. Establish the view transformation corresponding to a camera with its lower left pivot point in $(12,\ 14)$, rotated by an angle of $30°$ around this pivot and uniformly scaled by factor $\frac{1}{1.13}$ with a camera window in pixels of 15 width and 15 height.

3) Stack the former view transformation on top of the latter embedding transformation E, to matrix multiply these into one action matrix. Now calculate and visualise the subsequent arrow image as brought into the game window in GeoGebra.

Exercise 152 The Belgian UFO wave was a series of sightings of triangular UFOs over Belgium which lasted from November 1989 until April 1990. At a certain time, the pivot point P of such a triangular UFO was located in $(299, 99)$ when its nose point N was in $(300, 100)$. The military camera window measured 15 width by 10 height with the bottom left vertex C of the camera's rectangle situated in $($**298**$,\ 97)$ on the UFO's right wing tip as portrayed. This isosceles UFO is only principally portrayed, meaning its orientation and proportions may differ from this picture. The picture is solely provided for a better understanding of the above listed points P, N and C.



1) Given these conditions, answer the matrix product as such to deliver the view transformation which brings this UFO horizontally (meaning $\overrightarrow{PN}$ horizontally) into our game window.

2) Compute the former matrix product using GeoGebra to answer the UFO view transformation matrix.

3) Also append a window-filling scaling for a ground based view port of **60** width by **40** height. This scaling is uniform. Recompute the former matrix product now including the latter scaling using GeoGebra to answer the zoomed UFO view transformation matrix.

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